IDENTITIES BETWEEN FIELD-EQUATIONS IN THE GENERAL FIELD-THEORY OF SCHOUTEN AND VAN DANTZIG

and

CERTAIN RESULTS IN THE THEORIES OF LEGENDRE AND CONFLUENT HYPERGEOMETRIC FUNCTIONS.

- by -

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LIST OF PUBLISHED PAPERS

(All but the last three have been embodied in the second part of this thesis).

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(2) On some definite integrals involving Legendre Functions. (The Phil. Mag. December, 1934).

(3) On the value of $\int_{-1}^{1} Q_m(\mu)\, Q_n(\mu)\, d\mu$ (The Proc. Benares Math. Society. Vol. 13).

(4) On the summation of infinite series of Legendre Polynomials. (The Bulletin Calcutta Math.Society. Vol. XXIII, 1931).

(5) On the summation of infinite series of Legendre functions. (The Bull. Calcutta Math. Society. Vol. XXIII, No.3, 1931).

(6) On some infinite series of Legendre functions (The Bull. Calcutta Math. Society, Vol. XXIV, No.1, [1932]).

(7) On series and integrals involving Associated Legendre functions. (The Bull. Calcutta Math. Society. Vol. XXIV, No.4).

(8) On a system of Spherical Harmonics. (The Bull. Calcutta Math. Society. Vol. XXV, No.3.)

(9) On certain expansions of zero in series of Associated Legendre functions. (The Proc. Benares Math. Society. Vol, 13).

(10) On some results involving the k-function, a particular case of the confluent hypergeometric function. (The Bull. Calcutta Math. Society, Vol. XXIV, No.3, 1932).

(11) On some results involving the k-function, a particular case of the confluent hypergeometric function. (The Proc. Benares Math. Society. Vol. 14).

(12) On various recurrence formulae for $P_n^m(z)$, $Q_n^m(z)$ for unrestricted values of n, m and z. (The Proc. Benares Math. Society, Vol. 12 [1931]).

(13) On the Non-orthogonality of Legendre functions. (In conjunction with Dr S.C. Dhar). (The Bull. Calcutta Math. Soc. Vol. XXIV, No.4).

(14) On Bateman's polynomials. (Accepted for publication in the Proc. Benares Math. Society, Vol.16)

(15) A note on Bessel Functions. (The Bull. Calcutta Math. Society, Vol. 26.).

PREFACE

The general field-theory of Schouten and van Dantzig forms an important step in the solution of the unification problem of the gravitational and electromagnetic phenomena in Physics. This theory depends on the use of a projective geometry employing five homogeneous coordinates. In the first part of this thesis an attempt is made to make a contribution to this unified field-theory by finding the identical relations between the field-equations in this theory. We have also shown the connection between these identities and the identities found by Professor E.T. Whittaker between the field-equations of Einstein's general relativity.

In the second part of the thesis we first develop certain series and integral properties of Legendre functions in a direction, which has received little attention till now. The importance of the properties of k_n -functions developed towards the end of the second part may be seen from the following remark of Professor E.T. Whittaker in his well-known paper (The Bulletin of the American Mathematical Society, Volume 10, [1903-04,] page 133) in which he defines the function $W_{k,m}(z)$:

"There are other members of the family of functions $W_{k,m}(z)$ which have not been noticed, but which give promise of interesting properties. Among these may be mentioned the families of functions for which $m = 0$ and those for which $m = \frac{1}{2}$ ".

The k_n- functions considered correspond to the case $m = \frac{1}{2}$, the associated differential equation having arisen recently in the theory of turbulence in researches of W. Tollmien and Th. von Kármán and also in the wave-mechanical theory of the α-particles by Theodor Sexl.

I take this opportunity to express my sincere thanks to Professor E.T. Whittaker for the kind encouragement and facilities given to me during the preparation of the main portion of my thesis at Edinburgh. My thanks are also due to the King Edward Memorial Society of C.P. and Berar, Nagpur, India, for awarding me a research fellowship, which made it possible for me to carry on my research-work at the Edinburgh University.

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P A R T I

The identical relations between field-equations in the general field-theory of Schouten and van Dantzig.

CHAPTER I

Introduction

1. The problem of unified field-theory is to find a geometry which will represent both gravitational and electromagnetic phenomena just as the Riemannian geometry of general relativity represents gravitational phenomena alone. The most recent attempts depend on the use of a projective geometry employing five homogeneous coordinates. This is a step in the geometrisation of Physics. It may be said to have started with Faraday's lines and tubes of force and electricity filling all space. Then we have the attempts of Einstein, who from 1915 to 1933 has been occupied in finding out a geometry to study Physics. Weyl's attempts in 1917-18 and those of Eddington[1] in 1921 to bring macroscopic Physics into geometrical form may be mentioned here. The general field-theory is the direct descendent of the five dimensional theory of Kaluza and Klein. This theory has received a clear and detailed investigation at the hands of Schouten and van Dantzig and in this chapter we give a short account of the theory before we take up some fresh contribution to the theory in the following chapters. As will be shown, the theories of Veblen and Hoffmann and Einstein and Mayer are particular cases of this theory.

In Riemannian geometry of ordinary general relativity a unification of electromagnetic and gravitational phenomena is impossible. All unification theories make use in some way of a fifth coordinate in the local spaces. Now it seems impossible to give this fifth coordinate any physical meaning and this leads to the conception due to Veblen that local spaces are projective spaces and the five coordinates are the well-known homogeneous coordinates of projective geometry.

Using projective local spaces, pseudo-parallelism can be generalised into a mapping of local spaces on each other in a projective way. Cartan created for the first time projective geometries of a very general kind based on this principle. But these geometries are too general for relativity, since there has to be some kind of a metric. To get such a metric Veblen introduced in each local space a non-degenerate quadric and imposed the condition that this quadric is an invariant of the process of mapping local spaces on each other. If we take this quadric as the unit sphere, there is such analogy with Riemannian geometry. But there is a big difference. The contact point and the hyperplane at infinity are no longer invariants of the mapping process.

We can take the quadric as a unit-sphere, then the difference with the Riemannian geometry is that the contact-point and the hyperplane at infinity are no longer invariant.

2. Homogeneous coordinates

We describe the four-dimensional space-time world by five homogeneous coordinates,

$$(x^0, x^1, x^2, x^3, x^4) \qquad (2.1)$$

These five numbers define a point so that we have ∞^5 points. Two points x and y are said to be coincident if

$$\frac{y_0}{x_0} = \frac{y^1}{x^1} = \frac{y^2}{x^2} = \frac{y^3}{x^3} = \frac{y^4}{x^4} \quad \text{- - - - - - - - - -} \quad (2.2)$$

A set of coincident points defines a <u>spot</u>. A <u>spot</u> is physically a point-event in space-time. Subject the coordinates to a transformation

$$x^{\nu'} = x^{\nu'}(x^0, x^1, x^2, x^3, x^4), \; (\nu' = 0, 1, 2, 3, 4) \quad \text{- - - -} \quad (2.3)$$

where these new coordinates are homogeneous functions of degree one, not necessarily linear, of the old coordinates. A group of these transformations is called H_5.

We also consider transformations of points,

$$\bar{x}^{\nu} = \rho x^{\nu} \qquad (\nu = 0, 1, 2, 3, 4 \quad) \quad \text{- - - -} \quad (2.4)$$

where ρ is an arbitrary function of x^{ω}, homogeneous and of degree zero. These last transformations leave invariant every spot. We denote the group of these transformations by F. These are peculiar to the projective theory of relativity. It gives change from point to point.

Tensors of Einstein's general Relativity are now called <u>affinors</u> because they belong to the affine geometry. We now introduce the analogues of tensors, which we call "<u>projectors</u>". A function of x^0, x^1, x^2, x^3, x^4, which is invariant under the transformation of the group H_5, and which acquires a factor ρ^r under transformation F is called a <u>scalar</u> of degree r.

A set of five functions, which under H_5 have the law

$$V^{\nu'} = \sum_{\nu} \frac{\partial x^{\nu'}}{\partial x^{\nu}} V^{\nu} \quad \text{- - - - -} \quad (2.5)$$

and under F acquire the factor ρ^r so that

$$\bar{V}^\nu = \rho^r V^\nu \quad ---- \quad (2.6)$$

constitute a contravariant projector of rank one and degree r or a contravariant point of degree r . This corresponds to a contravariant vector of the ordinary tensor-calculus.

A set of five functions, which under H_5 have the law of transformation

$$W_\nu' = \Sigma_\nu \frac{\partial x^\nu}{\partial x^{\nu'}} W_\nu \quad ---- \quad (2.7)$$

and under F acquire ρ^r is called a covariant point of degree r . Schouten and van Dantzig call it a covariant projector of valence one and degree r .

Similarly a projector of contravariant valence p and covariant valence q and degree r is a set of 5^{p+q} homogeneous functions of the coordinates of degree r , which transform under H_5 like a product of p contravariant and q covariant points and under F acquire ρ^r .

Let y^1, y^2, y^3, y^4 be the ordinary non-homogeneous coordinates in four-dimension space-time. The differentials dy^1, dy^2, dy^3, dy^4 are regarded as coordinates in another 4-dimensional space called the <u>tangent-space</u> at the point. In this local tangent-space we introduce a Euclidean (Minkowski) metric ($ds^2 = d\tau^2 - (dx^2 + dy^2 + dz^2)$). The theory of these tangent-spaces together with the underlying space constitutes a Riemannian metric. The Minkowski coordinates at any point (T, X, Y, Z) are functions of the general coordinates y^1, y^2, y^3, y^4 in terms of which we describe the whole space so that $ds^2 = \sum_{pq} g_{pq}\, dy^p\, dy^q$; $(p,q = 1,2,3,4)$

Similarly in the new theory at each spot of space-time we have a local tangent-space E_4^*, which is a projective space. To begin with, we define E_4^* attached to any spot P as a space having coordinates $\mathfrak{z}^0, \mathfrak{z}^1, \mathfrak{z}^2, \mathfrak{z}^3, \mathfrak{z}^4$ which undergo the transformation

$$\mathfrak{z}^{\nu'} = \sum_{\nu} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \mathfrak{z}^{\nu} \quad ---- \quad (2.8)$$

when the coordinates of the underlying space are subjected to transformation H_5. The contact points of degree zero, which exist at this spot P are in one-to-one correspondence with the points of the E_4^* and may therefore be identified with them.

By Euler's theorem on homogeneous functions

$$x^{\nu'} = \sum_{\nu} \frac{\partial x^{\nu'}}{\partial x^{\nu}} x^{\nu} \quad ---- \quad (2.9)$$

Hence $(x^0, x^1, x^2, x^3, x^4)$ are not only homogeneous coordinates of a point of space-time but also coordinates of a point of degree one of the E_4^*. The corresponding point of the E_4^* identified with the point x^ν of the space-time is called the point of contact of the E_4^*.

3. The fundamental quadric

In every local space E_4^* let us introduce a quadric (3-dimensional quadratic hypersphere ϕ) which does not contain the point of contact x^ν,

$$\sum_{\mu\nu} G_{\mu\nu} V^\mu V^\nu = 0 \; ; \; \mathrm{Det}\,(G_{\mu\nu}) \neq 0 \; ; \quad ----- \quad (3.1)$$

$G_{\mu\nu}$ is a projector of covariant valence 2. So we normalize $g_{\mu\nu}$ by the condition that

$$\sum_{\mu\nu} G_{\mu\nu}\, x^{\mu}\, x^{\nu} = -\omega^2 \qquad ---- \quad (3.2)$$

where ω is a positive constant of dimensions of length. $\omega = \frac{k\hbar}{c}$, k being constant of gravitation, $\hbar = \frac{h}{2\pi}$, c is the velocity of light, and $\mu,\nu = 0, 1, 2, 3, 4$ and h is the Planck constant

In Einstein's theory

$$\sum_{ij} g_{ij}\, dx^i\, dx^j \qquad (i,j = 1, 2, 3, 4)$$

gives ten g's determining the gravitational-field. Here we have fifteen G's of which fourteen are at our disposal since one is normalized. This supplies ten G's to account for the gravitational field, and four for electromagnetic properties. As in Riemannian geometry using the $G_{\mu\nu}$'s to raise and lower the indices we get,

$$V_\lambda = \sum_{\mu} G_{\lambda\mu}\, V^{\mu} \qquad ---- \quad (3.3)$$

The point of contact x^λ will have a polar hyperplane with respect to the fundamental quadric

$$\left.\begin{aligned} & x_\lambda = \sum_{\mu} G_{\lambda\mu}\, x^{\mu} \\ \text{so} \quad & \sum_{\lambda} x_\lambda\, x^\lambda = -\omega^2 \\ \text{or} \quad & \sum_{\lambda} q_\lambda\, q^\lambda = -1 \\ \text{writing} \quad & x^\lambda = \omega\, q^\lambda \text{ and } x_\lambda = \omega\, q_\lambda \end{aligned}\right\} \qquad ---- \quad (3.4)$$

As in elementary projective geometry, we can find a Euclidean metric in which the fundamental quadric φ becomes a hyper-sphere, the point of contact its centre, the polar hyperplane of the point of contact becoming the plane at infinity. Since the hyperplane at infinity is the plane $\sum_\lambda q_\lambda v^\lambda = 0$, in this metric the square of the element of the length is $dx^2 + dy^2 + \cdots$ where $X = \omega \cdot \frac{\sum_\lambda a_\lambda v^\lambda}{\sum_\lambda q_\lambda v^\lambda}$ and $Y = \omega \frac{\sum b_\lambda v^\lambda}{\sum_\lambda q_\lambda v^\lambda} \cdots\cdots$. The equation of the quadric is

$$\left(\sum_\lambda a_\lambda v^\lambda\right) + \left(\sum_\lambda b_\lambda v^\lambda\right)^2 + \cdots\cdots - \left(\sum_\lambda q_\lambda v^\lambda\right)^2 = 0$$

and we assume that the point of contact which is the centre of the hypersphere has its $X, Y, Z, \cdots$ all zero. Hence

$$\sum a_\lambda x^\lambda = 0, \quad \sum b_\lambda x^\lambda = 0, \cdots$$

This equation must be the same as

$$\sum_{\lambda\mu} G_{\lambda\mu} v^\lambda v^\mu = 0.$$

Hence, we must have

$$K\, G_{\lambda\mu} = a_\lambda a_\mu + b_\lambda b_\mu + \cdots - q_\lambda q_\mu$$

Since

$$\sum_{\lambda\mu} G_{\lambda\mu} x^\lambda x^\mu = -\omega^2 \qquad \text{by (3.4)}$$

$$K \sum_{\lambda\mu} G_{\lambda\mu} x^\lambda \cdot x^\mu = \sum_{\lambda\mu} a_\lambda \cdot a_\mu x^\lambda x^\mu + \cdots - \sum_{\lambda\mu} q_\lambda q_\mu x^\lambda x^\mu$$

$-K\omega^2 = -\omega^2$ and therefore $K = 1$ and $G_{\lambda\mu} = a_\lambda a_\mu + \cdots - q_\lambda q_\mu$

Denoting by l the distance in this metric from the point of contact to an arbitrary point V^λ in the E_4^*, we get

$$l^2 = \frac{\omega^2}{V^2} \sum_{\lambda\mu} \left(G_{\lambda\mu} + q_\lambda q_\mu\right) V^\lambda V^\mu \qquad ---- \quad (3.5)$$

where $V = -\sum_\lambda q_\lambda V^\lambda$ is called the weight of the contravariant point V^λ.

Excess. Suppose with some projector we perform on the points the transformation F, $\bar{x}^\omega = \rho x^\omega$ where ρ is a constant and then perform the transformation $x^{\omega'} = \rho^{-1} x^\omega$ of H_5. So the new coordinates of the new points are the same as the old coordinates of the old points. If the projector acquires the factor ρ^ϵ, then ϵ is called the excess of the projector. Suppose the projector is of degree r in contravariant valence t, covariant valence s. When we perform $\bar{x}^\omega = \rho x^\omega$ of F, the projector acquires ρ^r; when we perform $x^{\omega'} = \bar{\rho} x^\omega$ of H_5 the projector acquires ρ^{s-t}. Therefore $\rho^\epsilon = \rho^r . \rho^{s-t}$ and hence

$$\epsilon = r + s - t .$$

Unless the contrary is stated all projectors henceforward will be supposed to be of excess zero. The point of contact x^ω has $r = 1$, $s = 0$, $t = 1$ and so $\epsilon = 0$. $g_{\mu\omega}$ has $r = -2$, $s = 2$, $t = 0$ and hence $\epsilon = 0$. The derivative of a scalar is a vector only if the scalar is of excess zero.

Now introduce the description of the space-time world by four non-homogeneous coordinates y^1, y^2, y^3, y^4. These are homogeneous functions of zero degree of x^0, x^1, x^2, x^3, x^4. The space of (y^1, y^2, y^3, y^4) (ordinary space-time) is denoted by X_4. In this space as in Einstein's general relativity we introduce at each point a tangent space E_4. The coordinates in it are $d\varphi^1, dy^2, dy^3, dy^4$. We identify the points $(y^1, y^2, \varphi^3, \varphi^4)$ of the X_4 with the spots of the E_4. We write

$$A^k_\nu \equiv \frac{\partial \xi^k}{\partial x^\nu} \qquad \left.\begin{array}{l} k = 1, 2, 3, 4 \\ \nu = 0, 1, 2, 3, 4 \end{array}\right\} \quad ---- \quad (3.6)$$

This is called an affinor-projector since

$$A^{k'}_{\nu'} = \sum_{k\mu} \frac{\partial \varsigma^{k'}}{\partial \varsigma^k} \cdot \frac{\partial x^\mu}{\partial x^{\nu'}} A^k_\mu$$

To every covariant vector ω_k, A^k_μ sets up a correspondence with a covariant point by the equation

$$\omega_\mu = \sum_k A^k_\mu \, \omega_k .$$

By Euler's theorem on homogeneous functions

$$\sum_\mu x^\mu A^k_\mu = 0 \quad \text{or} \quad \sum_\mu q^\mu A^k_\mu = 0 \quad ---- \quad (3.7)$$

To the affinor projector A^k_μ let us associate a dually corresponding A^ν_k defined by

$$\left.\begin{array}{l} \sum_\nu A^\nu_j A^k_\nu = \delta^k_j \\ \sum_\nu q_\nu A^\nu_j = 0 \end{array}\right\} \quad ---- \quad (3.8)$$

There exists a one-to-one correspondence between covariant and contravariant points of weight and excess zero of E^*_4 and covariant and co~~v~~travariant vectors of E_4 respectively.

The Projector A_μ^ν.

Define A_μ^ν as

$$A_\mu^\nu = \sum_k A_\mu^k A_k^\nu \qquad ---- (3.9)$$

From the equations

$$\sum_\nu A_j^\nu A_\nu^k = \delta_j^k$$

$$\sum_\nu q_\nu A_j^\nu = 0$$

and $$\sum_k A_\mu^k A_k^\nu = A_\mu^\nu$$

we get in general

$$A_\mu^\nu = \delta_\mu^\nu + q^\nu q_\mu \qquad ---- (3.9a)$$

Hence

$$\sum_\mu q^\mu A_\mu^\nu = \sum_\mu q^\mu \left(\delta_\mu^\nu + q^\nu q_\mu\right) = q^\nu - q^\nu = 0$$

Similarly $$\sum_\nu q_\nu A_\mu^\nu = 0$$

Hence we get that if V^ν is a contravariant point of weight zero (and so corresponds to a vector) then

$$\sum_\nu A_\nu^\mu V^\nu = \sum_\nu \left(\delta_\nu^\mu + q^\mu q_\nu\right) V^\nu = V^\mu$$

Similarly

$$\sum_\mu A_\nu^\mu W_\mu = W_\nu$$ if W_ν is a covariant

point of weight zero. Thus contravariant and covariant points of weight zero (contravariant and covariant vectors) are unaltered by transvection with A^{μ}_{ν}.

4. Define the difference of two spots λU^{ν} and μV^{ν} by

$$\frac{V^{\nu}}{V} - \frac{U^{\nu}}{U} \qquad \text{- - - - (4.1)}$$

where V and U are their weights. This is a contravariant point of weight zero and so corresponds to a vector. In particular the difference of a spot V^{ν} and the spot of the point of contact is

$$\frac{V^{\nu}}{V} - q^{\nu} \qquad \text{since } \sum_{\nu} q_{\nu} q^{\nu} = -1$$

So the difference of the spot of V^{ν} and the spot of the point of contact corresponds to a vector r^k

$$\left.\begin{aligned} \text{where} \quad r^k &= \sum_{\nu} A^k_{\nu}\left(\frac{1}{V} V^{\nu} - q^{\nu}\right) = \frac{1}{V}\sum_{\nu} A^k_{\nu} V^{\nu} \\ \text{or conversely} \quad \frac{1}{V} V^{\nu} - q^{\nu} &= \sum_k A^{\nu}_k r^k \\ \text{or} \quad V^{\nu} &= V\left(\sum_k A^{\nu}_k r^k + q^{\nu}\right) \end{aligned}\right\} \text{- - - (4.2)}$$

These equations set up a one-to-one correspondecne between the points r^k of the local E_4 (r^k being a vector from the origin of the E_4 to the points in E_4) and those spots V^{ν} of the E^*_4, which do not lie in the hyperplane q_{λ}. The point of contact of the E^*_4 corresponds to the origin of E_4 and the straight lines through these points correspond. The spots in E^*_4, which lie in the hyperplane q_{λ} correspond to the points at infinity. Thus E^*_4 is identified with E_4.

The resolution of a projector into an affinor and q-factors :-

$$r^{k} = \frac{1}{V} \sum_{\lambda} A^{k}_{\lambda} V^{\lambda} \; ; \; V \sum_{k} A^{\nu}_{k} r^{k} = \sum_{k\lambda} A^{\nu}_{k} A^{k}_{\lambda} V^{\lambda} = \sum_{\lambda} A^{\nu}_{\lambda} V^{\lambda}$$

$$V^{\nu} = V \sum_{k} A^{\nu}_{k} r^{k} + V q^{\nu} = \sum_{\lambda} A^{\nu}_{\lambda} V^{\lambda} + V q^{\nu} \quad \text{----} (4.2a)$$

We often denote the affinor part by ${}'V^{\nu}$. Therefore

$V^{\nu} = {}'V^{\nu} + Vq^{\nu}$; ${}'V^{\nu} = \sum_{\lambda} A^{\nu}_{\lambda} V^{\lambda}$ being the affinor part. ---- (4.3)

In general a projector is an affinor when in every suffix its transvectant with q^{ν} or q_{ν} vanishes. Thus a projector $T^{\lambda}_{\mu\nu}$ is an affinor if

$$\left.\begin{aligned} \sum_{\lambda} q_{\lambda} T^{\lambda}_{\mu\nu} &= 0 \\ \sum_{\mu} q^{\mu} T^{\lambda}_{\mu\nu} &= 0 \\ \sum_{\nu} q^{\nu} T^{\lambda}_{\mu\nu} &= 0 \end{aligned}\right\} \quad \text{----} (4.4)$$

Any projector can be expressed as a sum of products of affinors and factors q^{ν}, q_{ν} . The affinor part of a projector will be denoted by adjoining a dash. It is obtained by remembering that an affinor is unaltered by transvection with A^{ν}_{μ} whereas q^{ν} and q_{ν} vanish when transvected with A^{ν}_{μ} . Thus the affinor part of $T_{\mu}{}^{\nu}$ is simply $\sum_{\rho\sigma} A^{\rho}_{\mu} A^{\nu}_{\sigma} T^{\sigma}_{\rho}$

The metric in the E_4.

Carry over the metric we introduced in the E_4^* to the E_4 by means of the correspondence set up above between E_4 and E_4^*. If l is the distance in the E_4^* from the point of contact to the point V^ν,

$$l^2 = \frac{\omega^2}{V^2} \sum_{\lambda\mu} (G_{\lambda\mu} + q_\lambda q_\mu) V^\lambda V^\mu \qquad \text{by (3.5)}$$

Write

$$G_{\lambda\mu} + q_\lambda q_\mu = g_{\lambda\mu} \quad \text{- - - - - - - - (4.5)}$$

so $$l^2 = \frac{\omega^2}{V^2} \sum_{\lambda\mu} g_{\lambda\mu} V^\lambda V^\mu$$

Now
$$\sum_\lambda q^\lambda g_{\lambda\mu} = \sum_\lambda q^\lambda (G_{\lambda\mu} + q_\lambda q_\mu) = q_\mu - q_\mu = 0$$

Therefore $g_{\lambda\mu}$ is an affinor. The corresponding true affinor is

$$g_{ij} = \sum_{\lambda\mu} A_i^\lambda A_j^\mu g_{\lambda\mu}$$

Substituting from (4.3)
$$l^2 = \omega^2 \sum_{ij} g_{ij} r^i r^j$$

Considering the immediate neighbourhood of the origin of the E_4 (the point of contact of the E_4^*), l can be written as ds.

Comparing the infinitesimal equation $d\xi^i = \sum_\nu A_\nu^i dx^\nu$ with the finite equation $Vr^i = \sum_\nu A_\nu^i V^\nu$

we see that $V r^i$ or (since we are dealing with immediate neighbourhood of the origin $V = -\sum_\nu V^\nu q_\nu = -\sum_\nu x^\nu q_\nu = w$) $w r^i$ reduces in the infinitesimal case to $d\xi^i$.

$$\left.\begin{array}{l} \text{so } l^2 = w^2 \sum_{ij} g_{ij} r^i r^j \quad \text{becomes } ds^2 = \sum_{ij} g_{ij} d\xi^i d\xi^j \\ \text{and } \quad ds^2 = \sum_{\lambda\mu} g_{\lambda\mu}\, dx^\lambda\, dx^\mu \end{array}\right\} \text{ - - - (4.6)}$$

We identify g_{ij} with Riemann fundamental tensor of general relativity. So g_{ij} specifies gravitational field and $G^{\lambda\mu}$ specifies the gravitational field and the electromagnetic field.

We readily prove that

$$\left.\begin{array}{l} g^{\lambda\mu} = G^{\lambda\mu} + q^\lambda q^\mu \\ \sum_\mu g_{\lambda\mu}\, g^{\mu\nu} = A_\lambda^\nu \\ \sum_j g_{ij}\, g^{jk} = \delta_i^k \end{array}\right\} \text{ - - - (4.7)}$$

Covariant differentiation of a projector (of excess zero) can be defined by the equations,

$$\left.\begin{array}{lll} \nabla_\mu p & = \dfrac{\partial p}{\partial x^\mu} & \text{when } p \text{ is a scalar} \\ \nabla_\mu V^\nu & = \dfrac{\partial V^\nu}{\partial x^\mu} + \sum_\lambda \Pi^\nu_{\lambda\mu}\, V^\lambda & \\ \nabla_\mu W_\lambda & = \dfrac{\partial W_\lambda}{\partial x^\mu} - \sum_\nu \Pi^\nu_{\lambda\mu}\, W_\nu & \end{array}\right\} \text{ - - - (4.8}$$

Veblen in his projective relativity supposes that

$$\Pi^\nu_{\lambda\mu} = \Pi^\nu_{\mu\lambda} \quad .$$

We do not suppose so here. $\Pi^{\nu}_{\lambda\mu}$ are a set of 125 functions of degree -1. They are not projectors just as in ordinary Einsteinian relativity the Christoffel symbols $\left\{ {}^{\lambda\,\mu}_{\nu} \right\}$ are not tensors. In fact

$$\left.\begin{aligned} \Pi^{\nu'}_{\lambda'\mu'} &= \sum_{\lambda\mu\nu} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \cdot \frac{\partial x^{\mu}}{\partial x^{\mu'}} \cdot \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Pi^{\nu}_{\lambda\mu} + \sum_{\nu} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \cdot \frac{\partial^2 x^{\nu}}{\partial x^{\lambda'} \partial x^{\mu'}} \\ \text{or} \quad &= \sum_{\lambda\mu\nu} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \cdot \frac{\partial x^{\mu}}{\partial x^{\mu'}} \cdot \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Pi^{\nu}_{\lambda\mu} - \sum_{\lambda\mu} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \cdot \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^{\lambda} \partial x^{\mu}} \end{aligned}\right\} \quad --- (4.9)$$

5. The bivector $q_{\lambda\mu}$:-

Define $q_{\lambda\mu}$ by

$$\left.\begin{aligned} q_{\lambda\mu} &= \tfrac{1}{2}\left(\frac{\partial q_{\mu}}{\partial x^{\lambda}} - \frac{\partial q_{\lambda}}{\partial x^{\mu}} \right) \\ &= \tfrac{1}{2}\left(\nabla_{\lambda} q_{\mu} - \nabla_{\mu} q_{\lambda} \right) \end{aligned}\right\} \quad --- (5.1)$$

Since $\sum q^{\lambda} q_{\lambda\mu} = 0$ --- (5.2)

$q_{\lambda\mu}$ is an affinor and it is skew ($q_{\lambda\mu} = -q_{\mu\lambda}$). Hence it is called a bivector. We identify it (save for a constant factor) with the electromagnetic bivector.

The Chrystoffel symbols:-

Write

$$\left\{ {}^{\nu}_{\lambda\mu} \right\} = \frac{1}{2} \sum_{\rho} G^{\nu\rho} \left(\frac{\partial G_{\mu\rho}}{\partial x^{\lambda}} + \frac{\partial G_{\lambda\rho}}{\partial x^{\mu}} - \frac{\partial G_{\lambda\mu}}{\partial x^{\rho}} \right) \quad --- (5.3)$$

and define

$$\Gamma^{k}_{ij} = \frac{1}{2} \sum_{t} g^{kt} \left(\frac{\partial g_{it}}{\partial \xi_j} + \frac{\partial g_{jt}}{\partial \xi_i} - \frac{\partial g_{ij}}{\partial \xi_t} \right) \quad --- (5.4)$$

Since $G_{\lambda\mu} = g_{\lambda\mu} - q_\lambda \cdot q_\mu$; after some reduction (5.3) comes out as

$$\left.\begin{aligned}\left\{ {\nu \atop \lambda\mu} \right\} = \sum_{ijk} A^{\nu}_{k} A^{i}_{\lambda} A^{j}_{\mu} \Gamma^{k}_{ij} + \sum_{k} A^{\nu}_{k} \frac{\partial^2 \xi^k}{\partial x^\lambda \partial x^\mu} - q_\lambda q^{\cdot\nu}_{\mu} \\ - q_\mu q^{\cdot\nu}_{\lambda} - \tfrac{1}{2} q^\nu \left(\frac{\partial q_\mu}{\partial x^\lambda} + \frac{\partial q_\lambda}{\partial x^\mu} \right)\end{aligned}\right] \quad (5.5)$$

$$\Pi^{\nu}_{\lambda\mu} = \left\{ {\nu \atop \lambda\mu} \right\} + a\, q^\nu q_{\lambda\mu} + b\, q_\lambda q^{\nu}_{\mu} + c\, q_\nu q^{\nu}_{\lambda} \quad \text{- - - (5.6)}$$

where a, b, c are constants to be determined by the condition that

$$\nabla_\mu G_{\lambda\omega} = 0 \quad \text{- - - - (5.7)}$$

(5.7) gives $(a + b) = 0$

Write $q - 1$ for b or $-a$ and $p - 1$ for c. } - - - - (5.7a)

(5.6) then becomes

$$\Pi^{\nu}_{\lambda\mu} = \left\{ {\nu \atop \lambda\mu} \right\} - (q-1)\, q^\nu q_{\lambda\mu} + (q-1)\, q_\lambda q^{\nu}_{\cdot\mu} + (p-1)\, q_\mu q^{\nu}_{\cdot\lambda} \quad (5.8)$$

In the symmetrical theories of Veblen, Hoffmann and Pauli, $p = 1$, $q = 1$. In the Einstein and Mayer's five-dimensional unified theory, when projectised, q is undetermined while $p = 0$. In Schouten and van Dantzig theory up to the end of 1932 $p = 4$, $q = 2$. In Schouten's 1933 theory $q = 2$, p is arbitrary but satisfies $q^2 - 2pq + 2p > 0$.

The projector $S_{\lambda\mu}^{\cdot\;\cdot\;\nu}$ is defined as

$$\left.\begin{aligned} S_{\lambda\mu}^{\cdot\;\cdot\;\nu} &= \frac{1}{2}\left(\Pi_{\lambda\mu}^{\nu} - \Pi_{\mu\lambda}^{\nu}\right) \\ &= -(q-1)\, q_{\lambda\mu}\, q^{\nu} + \frac{p-q}{2}\, q_{\lambda}\, q_{\mu}^{\cdot\nu} - \frac{p-q}{2}\, q_{\mu}\, q_{\lambda}^{\cdot\nu} \end{aligned}\right\} \quad \text{- - - - (5.9)}$$

This gives

$$\left.\begin{aligned} &\sum_{\nu\sigma} S_{\nu\sigma\lambda}\, q^{\nu} q^{\sigma} = 0 \\ &\sum_{\rho\nu} S_{\mu\rho\nu}\, q^{\rho} q^{\nu} = 0 \\ &\sum_{\rho\nu} S_{\rho\mu\nu}\, q^{\rho} q^{\nu} = 0 \end{aligned}\right\} \quad \text{- - - - (5.9a)}$$

6. Define projectors $P_{\cdot\lambda}^{\mu}$ and $Q_{\cdot\lambda}^{\nu}$ as

$$\left.\begin{aligned} P_{\cdot\lambda}^{\nu} &= \sum_{\mu} \Pi_{\lambda\mu}^{\nu} x^{\mu} + \delta_{\lambda}^{\nu} = -p\,\omega\, q_{\cdot\lambda}^{\nu} \\ Q_{\cdot\lambda}^{\nu} &= -\omega\, q\, q^{\nu}_{\cdot\lambda} \end{aligned}\right\} \quad \text{- - - - (6.1)}$$

(6.1) gives

$$\left.\begin{aligned} \nabla_{\mu}\, q^{\nu} &= -\, q\, q_{\cdot\mu}^{\nu} \\ \nabla_{\mu}\, q_{\nu} &= -\, q\, q_{\nu\mu} \\ \nabla_{\mu}\, A_{\lambda}^{\nu} &= -\, q\,(q_{\lambda} q_{\cdot\mu}^{\nu} + q^{\nu} q_{\lambda\mu}) \\ \nabla_{\mu}\, q_{\lambda\nu} &= -\, q\,(q_{\lambda} q_{\nu\mu} + q_{\nu} q_{\lambda\mu}) \end{aligned}\right\} \quad \text{- - - - (6.2)}$$

We determine an affine-connexion by means of a differential operator $\overset{R}{\nabla}_{\mu}$, which represents the affinor part of the projective covariant derivative of an affinor.

$$\overset{R}{\nabla}_\mu V^\nu = \sum_{\rho\sigma} A^\rho_\mu A^\nu_\sigma \nabla_\rho V^\sigma \quad , \text{ since } \sum_\nu V^\nu q_\nu = 0$$

$$\overset{R}{\nabla}_\mu w_\lambda = \sum_{\rho\pi} A^\rho_\mu A^\pi_\lambda \nabla_\rho w_\pi \quad , \text{ since } \sum_\nu w_\nu q^\nu = 0$$

The affinor (in latin letters) corresponding to $\overset{R}{\nabla}_\mu V^\nu$ is

$$\overset{R}{\nabla}_j V^k = \sum_{\rho\sigma} A^\rho_j A^k_\sigma \nabla_\rho V^\sigma$$

Similarly

$$\overset{R}{\nabla}_j w_i = \sum_{\rho\pi} A^\rho_j A^\pi_i \nabla_\rho w_\pi$$

We extend this affine connexion to a projective one by the additional conditions $\overset{R}{\nabla}_\mu q^\nu = 0$ and $\overset{R}{\nabla}_\mu q_\nu = 0$.
Let V^ν be any contravariant point of excess zero and so of degree one. Then

$$\left.\begin{aligned} &\overset{R}{\nabla}_\mu V^\nu = \frac{\partial V^\nu}{\partial x^\mu} + \sum_\lambda \overset{R}{\Pi}{}^\nu_{\lambda\mu} V^\lambda \\ &\text{where } \overset{R}{\Pi}{}^\nu_{\lambda\mu} = \left\{ {\nu \atop \lambda\mu} \right\} + q^\nu q_{\lambda\mu} - q_\lambda q^{\nu}_{\cdot\mu} - q_\mu q^{\nu}_{\cdot\lambda} \end{aligned}\right\} \quad \text{---- (6.3)}$$

are the parameters of the covariant differentiation of the Riemannian geometry written in homogeneous coordinates.
Applying this

$$\overset{R}{\nabla}_j g_{ik} = 0$$

Also

$$\overset{R}{S}{}_{\lambda\mu}{}^{\cdot\cdot\nu} = \tfrac{1}{2}\left[\overset{R}{\Pi}{}^\nu_{\lambda\mu} - \overset{R}{\Pi}{}^\nu_{\mu\lambda}\right] = q_{\lambda\mu} q^\nu$$

If F_{ij} is the electromagnetic six-vector, then

$F_{ij} = \frac{\partial \varphi_j}{\partial \varphi_i} - \frac{\partial \varphi_i}{\partial \varphi_j}$ where $\varphi_1, \varphi_2, \varphi_3$ are the components of the Electro-magnetic potential vector, and $-\varphi_4$ is the electric potential. We identify F_{ij} as

$$F_{ij} = \frac{q_c}{k} q_{ij} \quad ---- \quad (6.4)$$

where c is the velocity of light and k is a constant of dimensions $M^{-\frac{1}{2}} L^{\frac{1}{2}}$.

(6.4) gives in homogeneous coordinates

$$F_{\lambda\mu} = \frac{q_c}{k} q_{\lambda\mu} \text{ that is } \frac{\partial \varphi_\mu}{\partial x^\lambda} - \frac{\partial \varphi_\lambda}{\partial x^\mu} = \frac{q_c}{2k}\left(\frac{\partial q_\mu}{\partial x^\lambda} - \frac{\partial q_\lambda}{\partial x^\mu}\right) \quad --- \quad (6.5)$$

where φ_λ is the electromagnetic potential.

In Einstein's general relativity we have the Riemann Tensor

$$K^{\cdots s}_{pqr} = \frac{\partial}{\partial x^r}\left\{ {pq \atop s} \right\} - \frac{\partial}{\partial x^q}\left\{ {pr \atop s} \right\} + \sum_t \left\{ {pq \atop t} \right\}\left\{ {tr \atop s} \right\} - \sum_t \left\{ {pr \atop t} \right\}\left\{ {qt \atop s} \right\}$$

This referred to homogeneous coordinates gives

$$K^{\cdots \omega}_{\omega\mu\lambda} = \frac{\partial \overset{R}{\Pi}{}^{\omega}_{\lambda\omega}}{\partial x^\mu} - \frac{\partial \overset{R}{\Pi}{}^{\omega}_{\lambda\mu}}{\partial x^\omega} + \sum_\rho \overset{R}{\Pi}{}^{\omega}_{\rho\mu} \overset{R}{\Pi}{}^{\rho}_{\lambda\omega} - \sum_\rho \overset{R}{\Pi}{}^{\omega}_{\rho\omega} \overset{R}{\Pi}{}^{\rho}_{\lambda\mu} \quad --- \quad (6.6)$$

From $\Pi^{\omega}_{\lambda\mu}$ we form the <u>projective curvature tensor</u>

$$N^{\cdots \omega}_{\omega\mu\lambda} = \frac{\partial \Pi^{\omega}_{\lambda\omega}}{\partial x^\mu} - \frac{\partial \Pi^{\omega}_{\lambda\mu}}{\partial x^\omega} + \sum_\rho \Pi^{\omega}_{\rho\mu} \Pi^{\rho}_{\lambda\omega} - \sum_\rho \Pi^{\omega}_{\rho\omega} \Pi^{\rho}_{\lambda\mu} \quad --- \quad (6.7)$$

From (6.6) and (6.7) we get

$$\left.\begin{aligned} N^{\cdot\cdot\cdot\nu}_{\omega\mu\lambda} - K^{\cdot\cdot\cdot\nu}_{\omega\mu\lambda} = {} & q\, q_\lambda \overset{R}{\nabla}_\mu q^{\cdot\nu}_{\cdot\omega} - q\, q^\nu \overset{R}{\nabla}_\mu q_{\lambda\omega} + p\, q_\omega \overset{R}{\nabla}_\mu q^{\cdot\nu}_{\cdot\lambda} \\ & - q\, q_\lambda \overset{R}{\nabla}_\omega q^{\cdot\nu}_{\cdot\mu} + q\, q^\nu \overset{R}{\nabla}_\omega q_{\lambda\mu} - p\, q_\mu \overset{R}{\nabla}_\omega q^{\cdot\nu}_{\cdot\lambda} \\ & + q^2 q^{\cdot\nu}_{\cdot\mu} q_{\lambda\omega} - 2p\, q_{\omega\mu} q^{\cdot\nu}_{\cdot\lambda} \\ & + p\,q \sum_\rho \Big(- q^\nu q_\omega q_{\rho\mu} q^{\cdot\rho}_{\cdot\lambda} + q_\lambda q_\mu q^{\cdot\nu}_{\cdot\rho} q^{\cdot\rho}_{\cdot\omega} + q^\nu q_\mu q_{\rho\omega} q^{\cdot\rho}_{\cdot\lambda} \\ & \qquad - q_\lambda q_\omega q^{\cdot\nu}_{\cdot\rho} q^{\cdot\rho}_{\cdot\mu} \Big) \end{aligned}\right\} \text{- - - - (6.8)}$$

Forming the generalized contracted Riemann tensor

$$\left.\begin{aligned} & N_{\mu\lambda} = \sum_\rho N^{\cdot\cdot\cdot\rho}_{\rho\mu\lambda}\,; \quad K_{\mu\lambda} = \sum_\rho K^{\cdot\cdot\cdot\rho}_{\rho\mu\lambda} \quad \text{and the scalars} \\ & N = \sum_{\mu\lambda} G^{\mu\lambda} N_{\mu\lambda} \quad \text{and} \quad K = \sum_{\lambda\mu} g^{\mu\lambda} K_{\mu\lambda} \end{aligned}\right\} \text{- - (6.9)}$$

we get

$$\left.\begin{aligned} N_{\mu\lambda} - K_{\mu\lambda} = {} & - q\, q_\lambda \sum_\rho \overset{R}{\nabla}_\rho q^{\cdot\rho}_{\cdot\mu} - p\, q_\mu \sum_\rho \overset{R}{\nabla}_\rho q^{\cdot\rho}_{\cdot\lambda} - p\,q\, q_\lambda q_\mu \sum_{\rho\sigma} q^{\rho\sigma} q_{\rho\sigma} \\ & + (pq - 2p - q^2) \sum_\rho q_{\rho\mu} q^{\cdot\rho}_{\cdot\lambda} \end{aligned}\right\} \text{(6.9a)}$$

and

$$N - K = (2pq - 2p - q^2) \sum_{\rho\sigma} q_{\rho\sigma}\, q^{\rho\sigma} \quad \text{- - - - (6.9b)}$$

7. The Variation Principle and the Field-Equations.

The simplest variational principle $\delta I = 0$ where

$$I = \int N \sqrt{G}\; dx^0\; dx^1\; dx^2\; dx^3\; dx^4 \quad \text{- - - - (7.1)}$$

where N is the projective scalar curvature, gives the field equations. In variation, $G_{\lambda\mu}$ are to be varied while the x^{ν} are to be kept constant. Working out the variation we get

$$\delta I = \sum_{\lambda\mu}\int\sqrt{G}\cdot\left[K_{\lambda\mu} - \tfrac{1}{2}K g_{\lambda\mu} + (q^2 - 2pq + 2p)\left\{\begin{matrix}\tfrac{1}{2}g_{\lambda\mu}\sum_{\rho\sigma} q_{\rho\sigma}\, q^{\rho\sigma} - 2\sum_{\rho} q_{\lambda}^{\cdot\rho} q_{\mu\rho} \\ + 2q_{\lambda}\sum_{\rho}\overset{R}{\nabla}_{\rho}\, q_{\mu}^{\cdot\rho}\end{matrix}\right\}\right]\times\delta G^{\lambda\mu}\,d\tau$$

where $d\tau = dx^0\, dx^1\, dx^2\, dx^3\, dx^4$

So the variational equations are

$$K_{\lambda\mu} - \tfrac{1}{2}K g_{\lambda\mu} + (q^2 - 2pq + 2p)\left\{\tfrac{1}{2}g_{\lambda\mu}\sum_{\rho\sigma} q_{\rho\sigma}\, q^{\rho\sigma} - 2\sum_{\rho} q_{\lambda}^{\cdot\rho} q_{\mu\rho} + 2q_{\lambda}\sum_{\rho}\overset{R}{\nabla}_{\rho}\, q_{\mu}^{\cdot\rho}\right\} = 0 \quad (7.3)$$

Subtract this from the equation obtained by interchanging λ and μ. we get

$$q_{\lambda}\sum_{\rho}\overset{R}{\nabla}_{\rho}\, q_{\mu}^{\cdot\rho} - q_{\mu}\sum_{\rho}\overset{R}{\nabla}_{\rho}\, q_{\lambda}^{\cdot\rho} = 0 \qquad (\lambda, \mu = 0, 1, 2, 3, 4)$$

These equations can only be satisfied if

$$\sum_{\rho}\overset{R}{\nabla}_{\rho}\, q_{\lambda}^{\cdot\rho} = 0 = B_{\lambda} \text{ say } (\lambda = 0, 1, 2, 3, 4) \qquad ---- \quad (7.4)$$

(7.3) then reduces to the equation symmetrical in λ and μ

$$K_{\lambda\mu} - \tfrac{1}{2}K g_{\lambda\mu} + (q^2 - 2pq + 2p)\left\{\tfrac{1}{2}g_{\lambda\mu}\sum_{\rho\sigma} q_{\rho\sigma}\, q^{\rho\sigma} - 2\sum_{\rho} q_{\lambda}^{\cdot\rho} q_{\mu\rho}\right\} = 0 \qquad -- \quad (7.5)$$

The Einstenian field-equations of Gravitation and Electromagnetism in empty space are

$$K_{ij} - \tfrac{1}{2}K g_{ij} = k E_{ij}$$

$k = 8\pi\mu$ where μ is the Newtonian gravitational potential due to one erg of matter at a distance of 1 cm.

where K_{ij} is the contracted Riemann Tensor
K is the scalar curvature
and E_{ij} is the energy tensor .

In empty space

$$E_{ij} = \frac{1}{4c^2} g_{ij} \sum_{kl} F_{kl} F^{kl} - \frac{1}{c^2} \sum_{k} F_{i}^{\cdot k} F_{jk}$$

where F_{pq} is the electromagnetic six-vector.

(If dx, dy, dz is the electric vector
hx, hy, hz is the magnetic vector

$F_{14} = \frac{1}{c} dx$; $F^{41} = c\,dx$, $F_{23} = \frac{1}{c} hx$, $F^{23} = c^3 hx$ and so on).

Thus the field equations are

$$\left.\begin{array}{l} K_{ij} - \frac{1}{2} K g_{ij} + \frac{k}{c^2}\left(\sum_{k} F_{i}^{\cdot k} F_{jk} - \frac{1}{4} g_{ij} \sum_{kl} F_{kl} F^{kl}\right) = 0 \\ \text{or in homogeneous coordinates} \\ K_{\lambda\mu} - \frac{1}{2} K g_{\lambda\mu} + \frac{k q^2}{k^2}\left(\sum_{\rho} g_{\cdot\lambda}^{\cdot\rho} g_{\mu\rho} - \frac{1}{4} g_{\lambda\mu} \sum_{\rho\sigma} g_{\rho\sigma} g^{\rho\sigma}\right) = 0 \end{array}\right\} \quad - - (7.6)$$

with help of (6.4) and (6.5).

This must be identified with (7.5). So we must have

$$-2\left(q^2 - 2pq + 2p\right) k^2 = k q^2 \qquad - - - - (7.7)$$

$$\left.\begin{array}{l} \text{(7.4) gives } \sum_{j} \overset{R}{\nabla_j} F_{i}^{\cdot j} = 0 \quad \text{or } \mathrm{Div}\left(F^{ij}\right) = 0 \\ \text{This is the Maxwell's tetrad of equations} \\ \frac{\partial dx}{\partial x} + \frac{\partial dy}{\partial y} + \frac{\partial dz}{\partial z} = 0 \,;\; -\frac{\partial dx}{\partial t} + \frac{\partial hz}{\partial y} - \frac{\partial hy}{\partial z} = 0 \quad \text{and so on} \end{array}\right\} \quad - - (7.8)$$

The other tetrad of Maxwell's equations are equivalent to the fact that $q_{\lambda\mu}$ is expressible as

$$q_{\lambda\mu} = \frac{1}{2}\left(\frac{\partial q_\lambda}{\partial x^\mu} - \frac{\partial q_\mu}{\partial x^\lambda}\right) \qquad \text{by (5.1)}$$

8. The Equations of Dirac

Dirac's numbers for projective theory are defined by the equations

$$\alpha^{(\lambda}\alpha^{\kappa)} = G^{\lambda\kappa} \tag{8.1}$$

and by using these numbers the Dirac-equation in Euclidian space can be written

$$\alpha^\mu\left(\frac{h}{i}\frac{\partial}{\partial_\mu} - \frac{e}{c}\varphi_\mu + mc\, q_\mu\right)\psi = 0 \tag{8.2}$$

In Riemannian space this equation is

$$\alpha^\mu\left(\frac{h}{i}\overset{R}{\nabla}_\mu - e/c\ \varphi_\mu + mc\, q_\mu\right)\psi = 0 \tag{8.3}$$

where

$$\left.\begin{aligned} \overset{R}{\nabla}_\mu &= \frac{\partial}{\partial x_\mu} + \overset{R}{\Lambda}_\mu \\ \overset{R}{\Lambda}_\mu &= -\frac{1}{4}\overset{R}{\Pi}{}^\kappa_{\mu\lambda}\alpha^\lambda\alpha_\kappa + \frac{1}{4}\alpha^\kappa\frac{\partial}{\partial x_\mu}\alpha_\kappa \end{aligned}\right\} \tag{8.4}$$

Now in projective theory we have to use instead of $\overset{R}{\nabla}_\mu$:

$$\left.\begin{aligned} \nabla_\mu &= \frac{\partial}{\partial x_\mu} + \Lambda_\mu \\ \Lambda_\mu &= -\frac{L}{4}\Pi^\kappa_{\mu\lambda}\alpha^\lambda\alpha_\kappa + \frac{L}{4}\alpha^\kappa\frac{\partial}{\partial x_\mu}\alpha_\kappa \end{aligned}\right\} \tag{8.5}$$

and the question arises whether the use of ∇_μ instead of $\overset{R}{\nabla}_\mu$ gives use to additional terms in the equations.

Now

$$\alpha^\mu\left(\nabla_\mu - \overset{R}{\nabla}_\mu\right) = \frac{L}{4}(p - 2q)\,\alpha^{[\mu\lambda k]}\, q_\lambda\, q_{\mu k} \qquad - - - - (8.6)$$

Hence, if we impose the condition that <u>the Dirac equation of projective relativity is identical with the ordinary Dirac equation and contains no additional terms</u> we have then

$$p - 2q = 0 \qquad - - - - (8.7)$$

9. <u>The variational equation in the case of a current.</u>

In ordinary theory the term with the current in the Maxwell equation (7.8), results, if we take instead of N a world function $\overset{0}{M} + K$ where

$$\overset{0}{M} = \sqrt{g}\;\frac{ik}{c}\;\frac{h}{i}\;\bar{\psi}\,\eta\,\alpha^\mu\,\overset{R}{\nabla}_\mu\,\psi \qquad - - - - (9.1)$$

This function is "practical real", that is, the imaginary part is a divergence. This is necessary because otherwise the result could not be real. In projective theory we have naturally to take instead of $\overset{0}{M}$

$$M = \sqrt{G}\;\frac{ik}{c}\;\frac{h}{i}\;\bar{\psi}\,\eta\,\alpha^\mu\,\nabla_\mu\,\psi \qquad - - - - (9.2)$$

Hence we have to impose the condition <u>that M is practical real</u> - - - - (9.3)

But this condition is identically satisfied if $\omega^2 > 0$. It is remarkable that for $\omega^2 < 0$ we find as a necessary and sufficient condition

$$p - 2q = 0$$

Hence (9.3) implies (8.7).

We now take $\omega > 0$ and $- - - - +$ as signature of $G_{\lambda k}$. The variational equation

$$\delta \int (M + N) \sqrt{G}\, dx^0\, dx^1\, dx^2\, dx^3\, dx^4 = 0 \qquad - - - - \quad (9.4)$$

gives

1. The equations of energy and impulse
   - of gravitation
   - of electromagnetic field
   - and of material waves

2. The equation of Maxwell

$$\overset{R}{\nabla}_j F_{i}^{\cdot j} = e s_i + \frac{p-2q}{2q} \frac{hk}{i} \overset{R}{\nabla}_k \left(\bar{\psi} \alpha_{[}{}^{k} \alpha_{i]} \alpha^0 \psi \right) - - - - \quad (9.5)$$

in which equation $e\, s_i$ is the current-vector,

$$s_\lambda \text{ being } = -s q_\lambda ; \quad (s = q^\lambda s_\lambda) \qquad - - - - \quad (9.6)$$

The additional term in (9.5) vanishes only if we impose the condition (8.7).

CHAPTER II

The Identities

1. Emmy Noether[1], by generalising certain results of Hilbert[2] has proved a general theorem to the effect that :

If F is a function of $\underline{n}$ quantities f, which are themselves functions of the $\underline{m}$ coordinates

$$\left(x^0, x^1, \cdots, x^{m-1}\right)$$

and their derivatives and if the integral

$$\int F \cdot dx^0 \cdot dx^1 \ldots dx^{m-1}$$

is invariant with respect to arbitrary transformations of the coordinates $\left(x^0, x^1, \cdots, x^{m-1}\right)$, then in the system of the $\underline{n}$ Lagrangian differential equations, which belong to the variational problem

$$\delta \int F\, dx^0 \cdots\, dx^{m-1} = 0$$

there are always $\underline{m}$, which are a consequence of the $\underline{n-m}$ others, in the sense, that between the $\underline{n}$ quantities f and their total differential coefficients with respect to $x^0, x^1, \cdots, x^{m-1}$, $\underline{m}$ linearly independent relations are identically satisfied. The best method of finding these identical relations in any particular problem is one due to Klein,[3] which is based on Lie's theory of infinitesimal transformations.

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(1) Göttinger Nachrichten, 1918, p. 236.
(2) Göttinger Nachrichten, 1915, p. 395.
(3) Göttinger Nachrichten, 1917, p. 469.

In this chapter let us try to work out the above theorem for the variational integral (7.1) of the last chapter, namely

$$\int N \sqrt{G}\, d\tau$$

where N is the projective scalar curvature and

$$d\tau = dx^0\, dx^1\, dx^2\, dx^3\, dx^4$$

By (7.2)

$$\delta \int N\sqrt{G}\, d\tau = \sum_{\lambda\mu} \int \sqrt{G} \cdot P_{\lambda\mu} \cdot \delta G^{\lambda\mu} \cdot d\tau \quad \ldots\ldots\ (1.1)$$

where

$$P_{\lambda\mu} \equiv \left[K_{\lambda\mu} - \frac{1}{2} K g_{\lambda\mu} + (q^2 - 2pq + 2p) \left\{ \frac{1}{2} g_{\lambda\mu} \sum_{\rho\sigma} q_{\rho\sigma} q^{\rho\sigma} + 2 q_\lambda \sum_\rho \overset{R}{\nabla}_\rho q_{\mu}^{\cdot\rho} - 2 \sum_\rho q_\lambda^{\cdot\rho} q_{\mu\rho} \right\} \right]$$

$$= 0 \quad \ldots\ldots\ (1.2)$$

give the field-equations.

2. <u>To find the value of $\delta G^{\lambda\mu}$.</u>

Comparing $G^{\lambda\mu}$ and $G^{\lambda\mu} + \delta G^{\lambda\mu}$, as they correspond to a transformation of coordinates, we have

$$G^{\lambda\mu} + \delta G^{\lambda\mu} = G^{\alpha\beta} \cdot \frac{\partial(x^\lambda + \delta x^\lambda)}{\partial x^\alpha} \cdot \frac{\partial(x^\mu + \delta x^\mu)}{\partial x^\beta}$$

$$= G^{\alpha\beta} \frac{\partial x^\lambda}{\partial x^\alpha} \cdot \frac{\partial x^\mu}{\partial x^\beta} + G^{\alpha\beta} \frac{\partial x^\lambda}{\partial x^\alpha} \cdot \frac{\partial(\delta x^\mu)}{\partial x^\beta} + G^{\alpha\beta} \frac{\partial x^\mu}{\partial x^\beta} \cdot \frac{\partial(\delta x^\lambda)}{\partial x^\alpha}$$

$$= G^{\lambda\mu} + G^{\lambda\beta} \cdot \frac{\partial(p^\mu)}{\partial x^\beta} + G^{\alpha\mu} \frac{\partial(p^\lambda)}{\partial x^\alpha}$$

where $p^{\lambda} = \delta x^{\lambda}$ and $p^{\mu} = \delta x^{\mu}$.

Hence

$$\delta G^{\lambda\mu} = G^{\lambda\beta} \frac{\partial(p^{\mu})}{\partial x^{\beta}} + G^{\alpha\mu} \frac{\partial(p^{\lambda})}{\partial x^{\alpha}}$$

This gives a comparison of the projector $G^{\lambda\mu}$ at $x^{\alpha} + \delta x^{\alpha}$ in the new coordinate-system with the value at x^{α} in the old coordinate-system. There would be no objection to using this value of $\delta G^{\lambda\mu}$ provided we took account of the corresponding $\delta(d\tau)$. We prefer to keep $d\tau$ fixed in comparison and must compare the values at x^{α} in both the systems. It is, therefore, necessary to subtract the change $\left\{ \delta x^{\alpha} \frac{\partial G^{\lambda\mu}}{\partial x^{\alpha}} \right\}$ of $G^{\lambda\mu}$ in the distance δx^{α}

Hence

$$\delta G^{\lambda\mu} = G^{\lambda\alpha} \frac{\partial(p^{\mu})}{\partial x^{\alpha}} + G^{\alpha\mu} \frac{\partial(p^{\lambda})}{\partial x^{\alpha}} - \frac{\partial G^{\lambda\mu}}{\partial x^{\alpha}} p^{\alpha} \tag{2.1}$$

where

$$p^{\alpha} = \delta x^{\alpha}$$

3. Using (2.2), (1.1) becomes

$$\int \sum_{\lambda\mu} \sqrt{G} \cdot P_{\lambda\mu} \cdot \delta G^{\lambda\mu} \, d\tau$$

$$= \sum_{\lambda\mu\alpha} \int \sqrt{G} \cdot P_{\lambda\mu} \left(G^{\lambda\alpha} \frac{\partial p^{\mu}}{\partial x^{\alpha}} + G^{\mu\alpha} \frac{\partial p^{\lambda}}{\partial x^{\alpha}} - \frac{\partial G^{\lambda\mu}}{\partial x^{\alpha}} \cdot p^{\alpha} \right) d\tau$$

We integrate this by parts, supposing the p's and their first and second derivatives to vanish at the boundary. We get

$$\sum_{\alpha\lambda\mu}\int\left[\frac{\partial}{\partial x^\alpha}\left\{P_{\lambda\mu}\sqrt{G}\cdot G^{\lambda\alpha}\right\}p^\mu+\frac{\partial}{\partial x^\alpha}\left\{P_{\lambda\mu}\sqrt{G}\cdot G^{\alpha\mu}\right\}p^\lambda+\frac{\partial G^{\lambda\mu}}{\partial x^\alpha}\,p^\alpha P_{\lambda\mu}\sqrt{G}\right]d\tau=0$$

Interchanging the dummy suffixes in each of the first two terms on the left we have

$$\sum_{\lambda\mu\alpha}\int\left[\frac{\partial}{\partial x^\mu}\left\{P_{\lambda\alpha}\sqrt{G}\cdot G^{\lambda\mu}\right\}+\frac{\partial}{\partial x^\lambda}\left\{P_{\alpha\mu}\sqrt{G}\cdot G^{\lambda\mu}\right\}+\frac{\partial G^{\lambda\mu}}{\partial x^\alpha}\,P_{\lambda\mu}\sqrt{G}\right]p^\alpha d\tau=0$$

Since p^α is arbitrary, its coefficients in this equation must vanish. Hence

$$\sum_{\lambda\mu}\left[\frac{\partial}{\partial x^\mu}\left\{P_{\lambda\alpha}G^{\lambda\mu}\sqrt{G}\right\}+\frac{\partial}{\partial x^\lambda}\left\{P_{\alpha\mu}\sqrt{G}\cdot G^{\lambda\mu}\right\}+\frac{\partial G^{\lambda\mu}}{\partial x^\alpha}P_{\lambda\mu}\cdot\sqrt{G}\right]=0 \qquad (\alpha=0,1,2,3,4)$$

Interchanging λ and μ in the second term we get

$$\sum_{\lambda\mu}\left[\frac{\partial}{\partial x^\mu}\left\{(P_{\lambda\alpha}+P_{\alpha\lambda})\sqrt{G}\cdot G^{\lambda\mu}\right\}+\frac{\partial G^{\lambda\mu}}{\partial x^\alpha}\,P_{\lambda\mu}\sqrt{G}\right]=0 \qquad (\alpha=0,1,2,3,4) \tag{3.1}$$

Substitute in (3.1)

$$\frac{\partial G^{\lambda\mu}}{\partial x^\alpha}=-\sum_\sigma\left\{{\mu \atop \alpha\sigma}\right\}G^{\lambda\sigma}-\sum_\sigma\left\{{\lambda \atop \alpha\sigma}\right\}G^{\mu\sigma} \tag{3.2}$$

A formula which can be proved in the same manner as the corresponding formula in ordinary tensor-analysis and put

$$X^{\mu\cdot}_{\cdot\alpha}=\frac{1}{2}\left(P^{\mu\cdot}_{\cdot\alpha}+P^{\cdot\mu}_{\alpha\cdot}\right) \qquad \ldots\ldots\ldots (3.3)$$

Then we have

$$\left[\sum_\mu\left[2\frac{\partial}{\partial x^\mu}\left(X^{\mu\cdot}_{\cdot\alpha}\right).\sqrt{G}+2X^{\mu\cdot}_{\cdot\alpha}\,\frac{\partial\sqrt{G}}{\partial x^\mu}\right]-\sqrt{G}\left[\sum_{\sigma\mu}\left\{{\mu \atop \alpha\sigma}\right\}P^{\sigma\cdot}_{\cdot\mu}+\left\{{\lambda \atop \alpha\sigma}\right\}P^{\cdot\sigma}_{\lambda\cdot}\right]\right]=0 \quad \ldots\ldots (3.4)$$

$$(\alpha=0,1,2,3,4)$$

Or interchanging the dummy suffixes μ and σ in the first term of the second [ ]- bracket and changing the dummy suffix λ into

μ and interchanging it with the dummy suffix σ in the second term of the second [ ] - bracket in (3.4) and remembering that

$$\frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial x^{\mu}} = \sum_{\sigma}\left\{{\sigma \atop \mu\,\sigma}\right\} \quad \cdots\cdots \text{(3.5)}$$

which too can be proved as in ordinary tensor-calculus, we get

$$2\sqrt{G}\cdot\sum_{\mu}\left[\frac{\partial}{\partial x^{\mu}}X^{\mu\cdot}_{\cdot\alpha} + \sum_{\sigma}\left\{{\sigma \atop \mu\,\sigma}\right\}X^{\mu\cdot}_{\cdot\alpha} - \sum_{\sigma}\left\{{\sigma \atop \alpha\,\mu}\right\}X^{\mu\cdot}_{\cdot\sigma}\right] = 0$$
$$(\alpha = 0,1,2,3,4)$$

or

$$\sum_{\mu}\left[\frac{\partial}{\partial x^{\mu}}\left(X^{\mu\cdot}_{\cdot\alpha}\right) + \sum_{\sigma}\left\{{\sigma \atop \mu\,\sigma}\right\}X^{\mu\cdot}_{\cdot\alpha}\right] - \sum_{\sigma\mu}\left\{{\sigma \atop \alpha\,\mu}\right\}X^{\mu\cdot}_{\cdot\sigma} = 0 \quad \cdots\cdots \text{(3.6)}$$
$$(\alpha = 0,1,2,3,4)$$

Now by means of (4.8) and (5.8) of Chapter I we have

$$\left.\begin{aligned}
&\text{(i)}\ \nabla_{\mu}X^{\mu\cdot}_{\cdot\alpha} = \frac{\partial X^{\mu\cdot}_{\cdot\alpha}}{\partial x^{\mu}} - \sum_{\sigma\mu}\Pi^{\sigma}_{\alpha\mu}X^{\mu\cdot}_{\cdot\sigma} + \sum_{\sigma\mu}\Pi^{\mu}_{\sigma\mu}X^{\sigma\cdot}_{\cdot\alpha}\\
&\text{(ii)}\ \Pi^{\mu}_{\sigma\mu} = \left\{{\mu \atop \sigma\,\mu}\right\}\\
&\text{(iii)}\ \Pi^{\sigma}_{\alpha\mu} = \left\{{\sigma \atop \alpha\,\mu}\right\} - (q-1)\,q^{\sigma}q_{\alpha\mu} + (q-1)\,q_{\alpha}q^{\sigma}_{\cdot\mu} + (p-1)\,q_{\mu}q^{\sigma}_{\cdot\alpha}
\end{aligned}\right\}\text{(3.7)}$$

Making use of (3.7 i) in (3.6) we have

$$\sum_{\mu\sigma}\left[\nabla_{\mu}X^{\mu\cdot}_{\cdot\alpha} + \Pi^{\sigma}_{\alpha\mu}X^{\mu\cdot}_{\cdot\sigma} + \left\{{\sigma \atop \mu\,\sigma}\right\}X^{\mu\cdot}_{\cdot\alpha} - \Pi^{\mu}_{\sigma\mu}X^{\sigma\cdot}_{\cdot\alpha} - \left\{{\sigma \atop \alpha\,\mu}\right\}X^{\mu\cdot}_{\cdot\sigma}\right] = 0 \quad \cdots \text{(3.8)}$$
$$(\alpha = 0,1,2,3,4)$$

Since

$$\sum_{\sigma\mu}\Pi^{\mu}_{\sigma\mu}X^{\sigma\cdot}_{\cdot\alpha} = \sum_{\sigma\mu}\left\{{\mu \atop \sigma\,\mu}\right\}X^{\sigma\cdot}_{\cdot\alpha} = \sum_{\sigma\mu}\left\{{\sigma \atop \mu\,\sigma}\right\}X^{\mu\cdot}_{\cdot\alpha}$$

we see that the third and the fourth terms in (3.8) cancel out. Then making use of (3.7 iii) we finally arrive at the identities between the field-equations in the form

$$\sum_{\mu\sigma}\left[\nabla_{\mu} X^{\mu\cdot}_{\cdot\alpha} - \left\{(q-1)\, q^{\sigma} q_{\alpha\mu} - (q-1)\, q_{\alpha} q^{\sigma}_{\cdot\mu} - (p-1)\, q_{\mu} q^{\sigma}_{\cdot\alpha}\right\} X^{\mu\cdot}_{\cdot\sigma}\right] = 0$$

$(\alpha = 0, 1, 2, 3, 4)$

. . . . (3.9)

or changing the dummy suffix σ into λ we have the required identities as

$$\sum_{\lambda\mu}\left[\nabla_{\mu} X^{\mu\cdot}_{\cdot\alpha} - \left\{(q-1)\, q^{\lambda} q_{\alpha\mu} - (q-1)\, q_{\alpha} q^{\lambda}_{\cdot\mu} - (p-1)\, q_{\mu} q^{\lambda}_{\cdot\alpha}\right\} X^{\mu\cdot}_{\cdot\lambda}\right] = 0$$

$(\alpha = 0, 1, 2, 3, 4)$

. . . .(3.10)

CHAPTER III

The verification of the identities.

1. By means of (6·9a), (6·9b) and (7.3) of Chapter I and (1.2) and (3.3) of Chapter II, we have

$$\left.\begin{aligned} X_{\lambda\mu} = N_{\lambda\mu} - \tfrac{1}{2} N G_{\lambda\mu} &+ (-q^2 + 3pq - 2p)\sum_\rho q_\lambda^{\cdot\rho}\, q_{\mu\sigma} \\ &-(p+q)\, q_\lambda \sum_\rho \overset{R}{\nabla}_\rho\, q_\mu^{\cdot\rho} \\ &+(q^2 - 2pq + 2p)\left[q_\lambda \sum_\rho \overset{R}{\nabla}_\rho\, q_\mu^{\cdot\rho} + q_\mu \sum_\rho \overset{R}{\nabla}_\rho\, q_\lambda^{\cdot\rho}\right] \end{aligned}\right\} \quad \cdots\cdots (1.1)$$

This can be written as

$$X_{\lambda\mu} = Z_{\lambda\mu} + (q^2 - 2pq + 2p)\, Y_{\lambda\mu} \quad \cdots\cdots (1.2)$$

where

$$Z_{\lambda\mu} = N_{\lambda\mu} - \tfrac{1}{2} N G_{\lambda\mu} + (-q^2 + 3pq - 2p)\sum_\rho q_\lambda^{\cdot\rho}\, q_{\mu\rho} - (p+q)\, q_\lambda \sum_\rho \overset{R}{\nabla}_\rho\, q_\mu^{\cdot\rho} \quad \cdots\cdots (1.3)$$

and

$$Y_{\lambda\mu} = q_\lambda \sum_\rho \overset{R}{\nabla}_\rho\, q_\mu^{\cdot\rho} + q_\mu \sum_\rho \overset{R}{\nabla}_\rho\, q_\lambda^{\cdot\rho} \quad \cdots\cdots (1.4)$$

Changing α into λ, λ into σ and substituting from (1.2), (3.10) of the last chapter breaks up into the sum of two parts, namely

$$\left.\begin{aligned} \sum_\mu \nabla_\mu Z_\lambda^{\cdot\mu} - (q-1)\sum_{\mu\sigma} Z_{\sigma\cdot}^{\cdot\mu}\, q^\sigma q_{\lambda\mu} + (q-1)\, q_\lambda \sum_{\sigma\mu} Z_{\sigma\cdot}^{\cdot\mu}\, q^{\sigma}_{\cdot\mu} \\ + (p-1)\sum_{\mu\sigma} Z_{\sigma\cdot}^{\cdot\mu}\, q_\mu\, q^{\sigma}_{\cdot\lambda} \end{aligned}\right\} \quad (1.5)$$

and

$$(q^2 - 2pq + 2p)\left[\begin{aligned} \sum_\mu \nabla_\mu Y^{\mu}_{\cdot\lambda} - (q-1)\sum_{\sigma\mu} Y^{\mu\cdot}_{\cdot\sigma}\, q^\sigma q_{\lambda\mu} + (q-1)\, q_\lambda \sum_{\sigma\mu} Y^{\mu\cdot}_{\cdot\sigma}\, q^{\sigma}_{\cdot\mu} \\ + (p-1)\sum_{\sigma\mu} Y^{\cdot\mu}_{\sigma\cdot}\, q_\mu\, q^{\sigma}_{\cdot\lambda} \end{aligned}\right] \quad (1.6)$$

Making use of the following relations, which we prove in the next sections,

$$\left.\begin{aligned}&\sum_{\mu} \nabla_{\mu}\left(N_{\lambda}^{\cdot\mu} - \tfrac{1}{2} N \delta^{\mu}_{\lambda}\right)\\ &\quad = (3pq - q^2 - 2p)\sum_{\rho\sigma} g_{\lambda\sigma} \nabla_{\rho} g^{\rho\sigma} + (q^2 - 2pq + 2p)\sum_{\mu\sigma} g^{\sigma\mu} \nabla_{\lambda} g_{\sigma\mu}\\ &\quad + q(p-q)\sum_{\rho\sigma} g^{\sigma\rho} \nabla_{\rho} g_{\sigma\lambda}\end{aligned}\right\} \quad (1.7)$$

$$\sum_{\mu\rho} \nabla_{\mu} \overset{R}{\nabla}_{\rho} g^{\mu\rho} = 0 \quad \text{- - - - - -} \quad (1.8)$$

$$\sum_{\sigma\mu} g^{\sigma\mu} \nabla_{\lambda} g_{\sigma\mu} = 2 \sum_{\rho\sigma} g^{\rho\sigma} \nabla_{\rho} g_{\lambda\sigma} \quad \text{- - - - - -} \quad (1.9)$$

in evaluating (1.5) we see that it reduces to

$$2(q^2 - 2pq + 2p)\sum_{\mu\rho} g_{\lambda\mu} \nabla_{\rho} g^{\mu\rho}$$

Similarly, after reduction (1.6) comes out to be equal to

$$2(q^2 - 2pq + 2p)\sum_{\mu\rho} g_{\mu\lambda} \nabla_{\rho} g^{\mu\rho}$$

So the left-hand side of (3.10) of the last chapter reduces to

$$\sum_{\mu\rho}\left[2(q^2 - 2pq + 2p) g_{\lambda\mu} \nabla_{\rho} g^{\mu\rho} + 2(q^2 - 2pq + 2p) g_{\mu\lambda} \nabla_{\rho} g^{\mu\rho}\right]$$

and this is zero since $g_{\mu\lambda} = - g_{\lambda\mu}$

Thus the identities are verified to be true by actual substitution.

2. To prove (1.7) we make use of the following relation given by Schonten and Van Dantzig.[1]

$$\sum_{\mu} \nabla_{\mu}\left[N_{\lambda}^{\cdot\mu} - \tfrac{1}{2} N \delta^{\mu}_{\lambda}\right]$$

$$= -\sum_{\sigma\mu} 2 S_{\lambda\sigma}^{\cdot\cdot\mu} N_{\mu\cdot}^{\cdot\sigma} - \sum_{\rho\sigma\mu} S_{\rho\sigma\mu} N_{\lambda}^{\cdot\mu\rho\sigma}$$

(1) Annals of Mathematics, Vol. 34, p. 293.

$$= -\sum_{\sigma\mu} 2\left[-(q-1)\, q_{\lambda\sigma}\, q^{\mu} + \frac{p-q}{2}\, q_{\lambda}\, q_{\sigma}^{\cdot\mu} - \frac{p-q}{2}\, q_{\sigma}\, q_{\lambda}^{\cdot\mu}\right] N_{\mu}^{\cdot\sigma}$$

$$- \sum_{\rho\sigma\mu}\left[-(q-1)\, q_{\rho\sigma}\, q_{\mu} + \frac{p-q}{2}\, q_{\rho}\, q_{\sigma\mu} - \frac{p-q}{2}\, q_{\sigma}\, q_{\rho\mu}\right] N_{\lambda}^{\cdot\mu\rho\sigma} \qquad \ldots\ldots(2.1)$$

remembering the values of $S_{\lambda\sigma}^{\cdot\cdot\mu}$, and $S_{\rho\sigma\mu}$ given by (5.9) of Chapter I.

The first term on the right of (2.1) contains transvections of the q's with $N_{\mu}^{\cdot\sigma}$. To evaluate these we make use of the following identities, which can be easily obtained from those given by Schouten and Van Dantzig in their paper in the <u>Annals of Mathematics</u>:-

$$\left.\begin{aligned}
\sum_{\rho} q^{\rho} N_{\rho\sigma} &= p \sum_{\rho} \nabla_{\rho}\, q^{\cdot\rho}_{\sigma} \\
\sum q^{\sigma} N_{\rho\sigma} &= q \sum_{\tau} \nabla_{\tau}\, q^{\cdot\tau}_{\rho} + q(p-q)\, q_{\rho} \sum_{\theta\tau} q^{\theta\tau}\, q_{\theta\tau} \\
\sum_{\sigma\rho} q^{\sigma\rho} N_{\rho\sigma} &= 0
\end{aligned}\right\} \qquad \ldots(2.2)$$

To evaluate the second term of (2.1) we observe from (6.8) of Chapter I, that

$$\begin{aligned}
N_{\lambda}^{\cdot\mu\rho\sigma} - K_{\lambda}^{\cdot\mu\rho\sigma} = &\sum_{\theta}\left[q\, q\, G^{\rho\mu\theta}\, \overset{R}{\nabla}_{\theta}\, q^{\sigma}_{\cdot\lambda} - q\, q^{\sigma}\, G^{\mu\theta}\, \overset{R}{\nabla}_{\theta}\, q^{\cdot\rho}_{\lambda}\right] \qquad (2.3) \\
&+ \sum_{\theta} p\, q_{\lambda}\, G^{\mu\theta}\, \overset{R}{\nabla}_{\theta}\, q^{\sigma\rho} - q\, q^{\rho}\, \overset{R}{\nabla}_{\lambda}\, q^{\sigma\mu} + q\, q^{\sigma}\, \overset{R}{\nabla}_{\lambda}\, q^{\rho\mu} \\
&- p\, q^{\mu}\, \overset{R}{\nabla}_{\lambda}\, q^{\sigma\rho} + q^{2}\, q^{\sigma\mu}\, q^{\rho}_{\cdot\lambda} \\
&- 2p\, q_{\lambda}^{\cdot\mu}\, q^{\sigma\rho} \\
&+ \sum_{\alpha} p\, q \left\{\begin{aligned} &-q^{\sigma}\, q_{\lambda}\, q^{\alpha\rho}\, q^{\mu}_{\alpha\cdot} + q^{\rho}\, q^{\mu}\, q^{\sigma\alpha}\, q_{\alpha\lambda} \\ &+ q^{\sigma}\, q^{\mu}\, q_{\alpha\lambda}\, q^{\rho\alpha} - q^{\rho}\, q_{\lambda}\, q^{\sigma}_{\cdot\alpha}\, q^{\alpha\mu} \end{aligned}\right\}
\end{aligned}$$

Transvection of the $K_{\lambda}^{\cdot\mu\rho\sigma}$ in (2.3) with the q's vanishes since $K_{\lambda}^{\cdot\mu\rho\sigma}$ is an affinor, when we substitute in the second term of (2.1) from (2.3). The transvections of the right-hand side of (2.3) with the q's can be easily evaluated by the known formulae of Chapter I. After these simplifications (2.1) comes out as

$$\sum_\mu \nabla_\mu \left(N_{\lambda}^{\cdot\mu} - \tfrac{1}{2} N \delta_\lambda^\mu \right)$$
$$= (3pq - q^2 - 2p) \sum_{\rho\sigma} g_{\lambda\sigma} \nabla_\rho g^{\rho\sigma} + (q^2 - 2pq + 2p) \sum_{\mu\sigma} g^{\sigma\mu} \nabla_\lambda g_{\sigma\mu}$$
$$+ q(p-q) \sum_{\rho\sigma} g^{\sigma\rho} \nabla_\rho g_{\sigma\lambda}$$

which is (1.7).

3. <u>To prove that</u> $\sum_{\mu\rho} \nabla_\mu \overset{R}{\nabla}_\rho g^{\mu\rho} = 0$

We have

$$\sum_\rho \overset{R}{\nabla}_\rho g^{\mu\rho} = \nabla_\rho g^{\mu\rho} + q g^\mu \sum_{\rho\sigma} g^{\rho\sigma} g_{\rho\sigma} \quad \cdots\cdots (3.1)$$

This comes out by remembering that

$$\overset{R}{\nabla}_\mu g^{\omega}_{\cdot\lambda} = \sum_{\pi\rho\sigma} (\delta_\lambda^\pi + g^\pi g_\lambda)(\delta_\mu^\rho + g^\rho g_\mu)(\delta_\sigma^\omega + g^\omega g_\sigma) \nabla_\rho g^{\sigma}_{\cdot\pi}$$

$$= \nabla_\mu g^{\omega}_{\cdot\lambda} + q g^\omega \sum_\rho g_{\rho\lambda} g^{\rho}_{\cdot\mu} + q g_\lambda \sum_\rho g^{\omega\rho} g_{\rho\mu} \quad \cdots\cdots (3.2)$$

which follows after some simplification.

In getting (3.2) we require a formula

$$\sum_\rho g^\rho \nabla_\rho g^{\omega}_{\cdot\lambda} = 0 \quad \cdots\cdots (3.3)$$

which can be easily seen to be true because

$$\sum_\rho g^\rho \nabla_\rho g^{\omega}_{\cdot\lambda} = \omega^{-1} \sum_\rho x^\rho \frac{\partial g^{\omega}_{\cdot\lambda}}{\partial x^\rho} - \sum_{\rho\sigma} g^\rho \Pi_{\lambda\rho}^\sigma g^{\omega}_{\cdot\sigma} + \sum_{\rho\sigma} g^\rho \Pi_{\sigma\rho}^\omega g^{\sigma}_{\cdot\lambda}$$

The first term on the right vanishes and

$$\left.\begin{aligned} \sum_\rho \Pi_{\lambda\rho}^\sigma g^\rho &= \omega^{-1}\left(P^{\sigma}_{\cdot\lambda} - \delta^\sigma_\lambda\right) \\ \sum_\rho \Pi_{\sigma\rho}^\omega g^\rho &= \omega^{-1}\left(P^\omega_\sigma - \delta^\omega_\sigma\right) \end{aligned}\right\} \quad \text{give}$$

$$\sum_\rho g^\rho \nabla_\rho g^{\nu}_{\cdot\lambda} = -\sum_\sigma \omega^{-1}(P^{\cdot\sigma}_{\cdot\lambda} - \delta^\sigma_\lambda)\, g^{\cdot\nu}_{\cdot\sigma} + \sum_\sigma \omega^{-1}(p^{\cdot\nu}_{\cdot\sigma} - \delta^\nu_\sigma)\, g^{\cdot\sigma}_{\cdot\lambda}$$

$$= 0$$

We also require for (3.2)

$$\left.\begin{aligned} \sum_\rho g^\rho \nabla_\mu g^{\nu}_{\cdot\rho} &= g \sum g^{\nu}_{\cdot\rho}\, g^{\rho}_{\cdot\mu} \quad \cdots\cdots (3.4) \\ \sum_\sigma g_\sigma \nabla_\mu g^{\sigma}_{\cdot\lambda} &= g \sum_\rho g_{\rho\lambda}\, g^{\rho}_{\cdot\mu} \quad (3.5) \end{aligned}\right\}$$

both of which follow from $\sum_\rho g^\rho g^{\nu}_{\cdot\rho} = 0$

We get (3.1) from (3.2) by raising λ, changing it into $\lambda = \mu = \rho$ and putting μ for ν.

Operating on both sides of (3.1) by ∇_μ we have

$$\sum_{\mu\rho} \nabla_\mu \overset{R}{\nabla}_\rho g^{\mu\rho} = \sum_{\mu\rho} \nabla_\mu \nabla_\rho g^{\mu\rho} + \sum_\mu g\, g^\mu \nabla_\mu \left\{\sum_{\rho\sigma} g^{\rho\sigma} g_{\rho\sigma}\right\} \quad \cdots\cdots (3.6)$$

Both the terms on the right-hand side of (3.6) can be seen to be equal to zero, as

$$\sum_\mu g^\mu \nabla_\mu \sum_{\rho\sigma} g^{\rho\sigma} g_{\rho\sigma} = 0$$ by means of (3.3) and

$$\sum_{\mu\rho} \nabla_\mu \nabla_\rho g^{\mu\rho} = \sum_{\mu\rho\theta\tau} \nabla_\mu \nabla_\rho \left[\tfrac{1}{2} G^{\theta\mu} G^{\tau\rho}(\nabla_\theta g_\tau - \nabla_\tau g_\theta)\right]$$
$$= \sum_{\mu\rho\theta\tau}\left[\tfrac{1}{2} G^{\mu\theta} \nabla_\mu \nabla_\rho \nabla_\theta g^\rho - \tfrac{1}{2} G^{\tau\rho} \nabla_\rho \nabla_\mu \nabla_\tau g^\mu\right]$$
$$= 0$$

by interchanging μ and ρ and changing the dummy suffix θ into τ in the first term on the right.

4. <u>To prove that</u> $\sum_{\sigma\mu} g^{\sigma\mu} \nabla_\lambda g_{\sigma\mu} = 2\sum_{\rho\sigma} g^{\rho\sigma} \nabla_\rho g_{\lambda\sigma}$

$$\sum_{\sigma\mu} g^{\sigma\mu} \nabla_\lambda g_{\sigma\mu} = \sum_{\sigma\mu} g^{\sigma\mu}\left[\frac{\partial g_{\sigma\mu}}{\partial x^\lambda} - \sum_\rho \Gamma^\rho_{\sigma\lambda} g_{\rho\mu} - \sum_\rho \Gamma^\rho_{\mu\lambda} g_{\sigma\rho}\right]$$
$$= \sum_{\sigma\mu} g^{\sigma\mu} \frac{\partial g_{\sigma\mu}}{\partial x^\lambda} - \sum_{\rho\sigma\mu} 2\left\{{\rho \atop \mu\lambda}\right\} g_{\rho\sigma} g^{\mu\sigma} \quad \cdots\cdots (4.1)$$

by making use of relations in 5 of Chapter I.

$$2\sum_{\rho\sigma} g^{\rho\sigma} \nabla_\rho g_{\lambda\sigma}$$
$$=\sum_{\rho\sigma} 2 g^{\rho\sigma}\left[\frac{\partial g_{\lambda\sigma}}{\partial x^\rho} - \sum_\tau \Pi^\tau_{\lambda\rho} g_{\tau\sigma} - \sum_\tau \Pi^\tau_{\sigma\rho} g_{\lambda\tau}\right]$$
$$=\sum_{\rho\sigma} 2 g^{\rho\sigma} \frac{\partial g_{\lambda\sigma}}{\partial x^\rho} - 2\sum_{\tau\rho\sigma} g_{\tau\sigma} g^{\rho\sigma} \left\{ {\tau \atop \lambda\rho} \right\}$$

again by relations in the first chapter.

$$=\sum_{\rho\sigma} 2 g^{\rho\sigma} \frac{\partial g_{\lambda\sigma}}{\partial x^\rho} - 2\sum_{\mu\rho\sigma} g_{\rho\sigma} g^{\mu\sigma} \left\{ {\rho \atop \mu\lambda} \right\} \quad \ldots\ldots (4.2)$$

Now

$$2\sum_{\rho\sigma} g^{\rho\sigma} \frac{\partial g_{\lambda\sigma}}{\partial x^\rho}$$
$$=\sum_{\rho\sigma} g^{\rho\sigma} \frac{\partial}{\partial x^\rho}\left\{\frac{\partial g_\sigma}{\partial x^\lambda} - \frac{\partial g_\lambda}{\partial x^\sigma}\right\}$$
$$=\sum_{\rho\sigma} g^{\rho\sigma} \frac{\partial^2 g_\sigma}{\partial x^\rho \partial x^\lambda} \quad \ldots\ldots (4.3)$$

$$\sum_{\sigma\mu} g^{\sigma\mu} \frac{\partial g_{\sigma\mu}}{\partial x^\lambda} = \frac{1}{2}\sum_{\sigma\mu} g^{\sigma\mu} \frac{\partial^2 g_\mu}{\partial x^\lambda \partial x^\sigma} - \frac{1}{2}\sum_{\sigma\mu} g^{\sigma\mu} \frac{\partial^2 g_\sigma}{\partial x^\lambda \partial x^\mu}$$
$$= \frac{1}{2}\sum_{\rho\sigma} g^{\rho\sigma} \frac{\partial^2 g_\sigma}{\partial x^\lambda \partial x^\rho} + \frac{1}{2}\sum_{\rho\sigma} g^{\rho\sigma} \frac{\partial^2 g_\sigma}{\partial x^\lambda \partial x^\rho}$$
$$= \sum_{\rho\sigma} g^{\rho\sigma} \frac{\partial^2 g_\sigma}{\partial x^\rho \partial x^\lambda} \quad \ldots\ldots (4.4)$$

Hence

$$\sum_{\sigma\mu} g^{\sigma\mu} \nabla_\lambda g_{\sigma\mu} = 2\sum_{\rho\sigma} g^{\rho\sigma} \nabla_\rho g_{\lambda\sigma}$$

by means of (4.4) and (4.3)

CHAPTER IV

Connexion with identities between field-equations in Einstein's General Relativity.

1. Finding the true affinor corresponding to the left-hand side of (3.10) of Chapter II and putting it equal to zero we have

$$\sum_{j} \overset{R}{\nabla}_j Z^{\dot{j}}_{i\cdot} + 2(q^2 - 2pq + 2p) \sum_{jl} g_{ji} \overset{R}{\nabla}_l g^{jl} = 0 \quad (j, i, l = 1, 2, 3, 4) \qquad \ldots (1.1)$$

This must give us the identities between field-equations in Einstein's general theory of relativity.

It is easy to see from the last chapter that

$$\begin{aligned} Z^{\dot{j}}_{i\cdot} &= K^{\dot{j}}_{i\cdot} - \tfrac{1}{2} K \delta_i^j + \frac{k}{c^2}\left(\sum_{k=1}^{4} F_{i}^{\cdot k} F^{\dot{j}}_{\cdot k} - \tfrac{1}{4}\delta_i^j \sum_{k,l=1}^{4} F_{kl} F^{kl}\right) \\ &= A^{\dot{j}}_{i\cdot} \quad \text{say} \quad (i, j = 1, 2, 3, 4) \end{aligned} \qquad (1.2)$$

(1.1) then by means of (7.4) and (7.7) of Chapter I becomes

$$\sum_{j=1}^{4} \overset{R}{\nabla}_j A^{\dot{j}}_{i} + \frac{k}{c^2} \sum_{j=1}^{4} F_{ij} B^j = 0 \quad ; \quad (i = 1, 2, 3, 4)$$

$$\text{or} \sum_{j} (A^{\dot{j}}_{i\cdot})_{;j} + \frac{k}{c^2} \sum_{j=1}^{4} F_{i j}^{\cdot\cdot} B^j = 0 \quad ; \quad (i = 1, 2, 3, 4) \quad \ldots\ldots (1.3)$$

where $B^{\dot{j}} = \overset{R}{\nabla}_i F^{ji} = 0 \; (i,j = 1, 2, 3, 4)$ gives Maxwell's equations and $A^{\dot{j}}_{i\cdot} = 0$ gives Einstein's combined gravitational and electromagnetic field-equations.

2. Professor E. T. Whittaker at the end of his paper[1] on "Hilbert's world-function" gives the identities between field-equations of Einstein's general theory of relativity. He has assumed the existence of magnetic and electric currents and massive particles. When we assume that there are no currents and no massive particles his identities reduce to our identities (1.3) above.

Hence in vacuo the identities given by Professor Whittaker are

$$A_p = \sum_q B^q M_{pq} \qquad \text{- - - - - - - (2.1)}$$

where A_p represents the vectorial divergence of the symmetrical tensor ~~of~~ A_{pq} which is equal to

$$\gamma\left(K_{pq} - \tfrac{1}{2} g_{pq} K\right) + \tfrac{1}{2}\left(\tfrac{1}{4} g_{pq} \sum_{r,s} X_{rs} X^{rs} - \sum_s X_{qs} X_{p}^{\cdot s}\right);$$

$$X_{rs} = M_{rs} = \frac{\partial \Phi_r}{\partial x^s} - \frac{\partial \Phi_s}{\partial x^r} = F_{rs} \text{ above;}$$

$$B^q = \frac{1}{\sqrt{-g}} \sum_q \frac{\partial X^{pq}}{\partial x^q}$$ so that $B^q = 0$ gives Maxwell's equations, and γ is a constant inversely proportional to the Newtonian constant of gravitation. Taking $\gamma = -\frac{1}{2}\frac{c^2}{k}$, (2.1) and (1.3) can be at once seen to be the same.

3. J. M. Whittaker[2] also at the end of his paper gives such identities in general relativity, taking into account the wave-mechanical considerations. When we do not take into account the wave-mechanical considerations he has brought into the identities, we shall just see that his identities also reduce to (1.3) above.

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(1) Proc. Royal Society, London, A, Vol. 113 (1927) p. 496.
(2) "On the principle of least action"; the Proc. Royal Society of London, A, Vol. 121; 1928, pp. 543 - 557.

His identities in space free from electric and magnetic currents and matter, and also free from the wave-mechanical restrictions become

$$\sum_{\nu} \left(A^{\cdot\nu}_{\mu}\right)_{\nu} - \sum_{\nu} B^{\nu} X_{\mu\nu} = 0 \qquad (\mu, \nu = 1,2,3,4) \qquad \ldots\ldots (3.1)$$

where

$$\tfrac{1}{2} A^{\mu\nu} = \gamma\left(K^{\mu\nu} - \tfrac{1}{2} K g^{\mu\nu}\right) + \tfrac{1}{2} E^{\mu\nu}$$

$$B^{\mu} = \sum_{\nu} \left(X^{\mu\nu}\right)_{\nu} = \sum_{\nu} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\nu}} \left(X^{\mu\nu}\sqrt{-g}\right)$$

$$E^{\mu\nu} = \sum_{\alpha\beta} \tfrac{1}{4} g^{\mu\nu} X_{\alpha\beta} X^{\alpha\beta} - \sum_{\alpha} X^{\mu\alpha} X^{\nu}{}_{\alpha}$$

and $X_{\mu\nu} = \dfrac{\partial \varphi_{\nu}}{\partial x^{\mu}} - \dfrac{\partial \varphi_{\mu}}{\partial x^{\nu}}$, φ_{μ} being the electromagnetic potential.

Taking $\gamma = -\frac{1}{2}\frac{c^2}{k}$; (3.1) can at once be seen to be the same as (1.3) above.

4. W. Pauli[1] also in his unified field-theory gives the following identities between field equations:-

$$\sum_{k} K^{\cdot k}_{i;k} - \varepsilon \sum_{k} X_{ik} K^{k}_{\cdot(0)} \equiv 0 \qquad (4.1)$$

where

$$K_{ij} = R_{ij} - \tfrac{1}{2} g_{ij} R + \frac{k}{c^2}\left(\sum_{r} F_{i}^{\cdot r} F_{jr} - \sum_{rs} \tfrac{1}{4} g_{ij} F^{rs} F_{rs}\right)$$

$$K^{i}_{\cdot(0)} = -\tfrac{1}{r} \frac{\sqrt{k}}{c} F^{ik}_{;k}$$

$$X_{ik} = r \frac{\sqrt{k}}{c} F_{ik}$$

and $\varepsilon = +1$

Pauli's K_{ij} is A_{ij} of (1.2) and his R_{ij} is our K_{ij}. Thus (4.1) in our notation becomes

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1. Ann. der Ph. 18. (1933) pp. 305 - 372. "Über die Formulierung der Naturgesetze mit fünf homogenen Koordinaten"

$$\sum_k (A^{k}_{i\cdot})_{;k} + \sum_{kj} \frac{k}{c^2} F_{ik} F^{kj}_{\cdot\,;j} = 0$$

or

$$\sum_j (A_{i\cdot}^{\,j})_{;j} + \frac{k}{c^2} \sum_j F_{ij} B^j = 0 \qquad \text{- - - - - (4.2)}$$

where $B^j = \sum_k F^{jk}{}_{;k} = \sum_k \overset{R}{\nabla}_k F^{jk} = \sum_i \overset{R}{\nabla}_i F^{ji}$

Thus we see that (4.2) is the same as (1.3).

CHAPTER V

Introduction of λ and the current-vector into the identities.

1. Let us first try to introduce λ, the universal constant proportional to the square of curvature of the world, into the field-equations and the identities given in the first two chapters.

If in the variational integral (7.1) of chapter I we take $N - 2\lambda$ for N and proceed in the same manner we arrive in place of (7.5) at

$$K_{\lambda\mu} - \tfrac{1}{2} k\, g_{\lambda\mu} + \lambda g_{\lambda\mu} + (q^2 - 2pq + 2p)\left\{\tfrac{1}{2} g_{\lambda\mu} \sum_{\rho\sigma} g_{\rho\sigma}\, g^{\rho\sigma} - 2 \sum_{\rho} g_{\lambda}^{\cdot\rho}\, g_{\mu\rho}\right\} = 0 \quad (1.1)$$
$$(\lambda, \mu = 0, 1, 2, 3, 4)$$

$X_{\lambda\mu}$ of the identities (3.10) of Chapter II now takes the form

$$X_{\lambda\mu} = Z_{\lambda\mu} + \lambda g_{\lambda\mu} + (q^2 - 2pq + 2p)\, Y_{\lambda\mu} \quad \text{------} (1.2)$$

in place of (1.2) of Chapter III.

The identities (3.10) of Chapter II now take the following form

$$\sum_{\lambda\mu}\left[\nabla_\mu X^{\mu\cdot}_{\cdot\alpha} - \left\{(q-1)\, q^{\lambda}\, g_{\alpha\mu} - (q-1)\, q_\alpha\, q^{\lambda}_{\cdot\mu} - (p-1)\, q_\mu\, q^{\lambda}_{\cdot\alpha}\right\} X^{\mu\cdot}_{\cdot\lambda}\right] = 0$$
$$(\alpha = 0, 1, 2, 3, 4) \quad \ldots\ldots (1.3)$$

where $X_{\lambda\mu}$ is given by (1.2) above.

Finding the true affinor corresponding to the left-hand side of (1.3) we have

$$\sum_j \overset{R}{\nabla}_j\, \overline{Z}^{j}_{i\cdot} + 2\,(q^2 - 2pq + 2p) \sum_{jl} q_{ji}\, \overset{R}{\nabla}_l\, q^{jl} = 0 \quad \text{----} \quad (1.4)$$
$$(j, i, l = 1, 2, 3, 4)$$

where $\bar{Z}_{ij} = K_{ij} - \frac{1}{2} K g_{ij} + \lambda g_{ij}$

$$+ (q^2 - 2pq + 2p)\left\{\frac{1}{2} g_{ij} \sum_{kl=1}^{4} g_{kl}\, g^{kl} - 2 \sum_{l=1}^{4} g_{i}^{\cdot l}\, g_{jl}\right\}$$

$$= K_{ij} - \frac{1}{2} K g_{ij} + \lambda g_{ij}$$

$$+ \frac{k}{c^2}\left(\sum_{k=1}^{4} F_{i\cdot}^{\cdot k} F_{jk} - \frac{1}{4} g_{ij} \sum_{kl=1}^{4} F_{kl} F^{kl}\right)$$

$$= \bar{A}_{ij} \qquad \text{say}$$

(1.4) can be written as

$$\sum_{j} \overset{R}{\nabla}_j \bar{A}_{i\cdot}^{\cdot j} + \frac{k}{c^2} \sum_{j} F_{ij}\, B^{j} = 0$$

or $\sum_{j} (\bar{A}_{i\cdot}^{\cdot j})_{,j} + \frac{k}{c^2} \sum_{j} F_{ij}\, B^{j} = 0$ - - - - - (1.5)

where $\bar{A}_{ij} = 0$ gives Einstein's combined gravitational and electromagnetic field-equations with λ introduced into them and $B^{j} = \sum_{i} \overset{R}{\nabla}_i F^{ji} = 0$ gives Maxwell's equations $(i, j = 1, 2, 3, 4)$

2. The term with the current-vector in the Maxwell's equations has been introduced in (9.4) of Chapter I when we take $p = 2q$

(1.5) in this case of $p = 2q$ can be written

$$\sum_{j} (\bar{A}_{i\cdot}^{\cdot j})_{,j} + \sum_{j} \frac{k}{c^2} F_{ij}\, \bar{B}^{j} = 0 \qquad \text{- - - - - (2.1)}$$

where $\bar{B}^{j} = \overset{R}{\nabla}_i F^{ij} - e s^{j} = 0$ - - - - - (2.2)

gives the Maxwell's equation with the current vector es_j introduced.

It is easy to see that (1.5) can be put into the form (2.1) since

$$\sum_{j} F_{ij}\, s^{j} = 0 \qquad \text{- - - - - (2.3)}$$

(2.3) may be seen to be true by converting it into homogeneous coordinates.

In homogeneous coordinates the left-hand side of (2.3) becomes

$$-\sum_{\mu} F_{\lambda\mu} \cdot e \cdot s \, q^{\mu}$$

$$\text{or} \quad -\frac{q e s c}{k} \sum_{\mu} q_{\lambda\mu} \, q^{\mu} \tag{2.4}$$

and this is zero by (5.2) of Chapter I.

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PART II

Certain results involving Legendre functions and the Kn-functions, a particular case of the confluent hyper-geometric functions.

Chapter I

Series and integrals involving Legendre Functions.

Paper I

On some definite integrals involving Legendre functions

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1. A few definite integrals involving more than two Legendre functions in the integrand have been considered by Ferrers, Adams, Dougall, Nicholson and Bailey. We take for example the following integrals.

[1] $$(1 \cdot 1) \int_{-1}^{1} P_p(\mu)\, P_q(\mu)\, P_r(\mu)\, d\mu,$$

[2] $$(1 \cdot 2) \int_{-1}^{1} (1-\mu^2)^{\frac{1}{2}m} P_p^m(\mu)\, P_q^m(\mu)\, d\mu \quad \text{or}$$

[3] $$\int_{0}^{1} (1-\mu^2)^{\frac{1}{2}m} P_p^m(\mu)\, P_q^m(\mu)\, d\mu,$$

[4] $$(1 \cdot 3) \int_{-1}^{1} P_\alpha(\mu)\, P_\beta(\mu)\, Q_\gamma(\mu)\, d\mu \quad \text{and}$$

[2] $$(1 \cdot 4) \int_{-1}^{1} \{P_p(\mu)\}^4\, d\mu.$$

It will be interesting to consider other definite integrals of the above type with three Legendre functions in the integrand, various possible combinations of the Legendre functions of the two kinds being taken into account. We also generalize for unrestricted values of m and n the definite integrals

$$\int_{-1}^{1} P_n(\mu)\, Q_m(\mu)\, d\mu,$$

$$\int_{-1}^{1} Q_n(\mu)\, d\mu \quad \text{and}$$

$$\int_{0}^{1} P_n(\mu)\, Q_m(\mu)\, d\mu,$$

considered by Nicholson[5] for integer-values of n and m only.

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[1] Hobson, *Spherical and Ellipsoidal Harmonics*, p. 87. The references to Ferrers and Adams, who considered this integral, are given there.

[2] W. N. Bailey, "On the product of two Legendre functions," *Proc. Camb. Phil. Soc.* 29 (1933), 173, results (4·1) and (4·2).

[3] J. Dougall, *Proc. Edin. Math. Soc.* 37 (1919), 33-47, formula (3).

[4] Nicholson, *Phil. Mag.* (6) 43 (1922), 783.

[5] *Phil. Mag.* (6), 43 (1922), 1-29, "Zonal Harmonics of the second kind."

2. In the expansion for $(\mu^2-1)^{\frac{1}{2}m} P_p^m(\mu)\cdot Q_q^m(\mu)$ given by W. N. Bailey[1] we put $m=0$. We have the result

$$(2\cdot 1)\quad P_p(\mu)\,Q_q(\mu)=\sum_{r=0}^{p}\frac{A_r\,A_{p-r}\,A_{q-p+r}}{A_{q+r}}\,\frac{2q-2p+4r+1}{2q-2p+2r+1}\,Q_{q+p+2r}(\mu)$$

[2] where $\dfrac{(\frac{1}{2})_s}{(s+m)!}=A_s^m$, *i.e.*, $\dfrac{(\frac{1}{2})_{q-r}}{(q+m-r)!}=A_{q-r}^m$; $A_{r,m}=\dfrac{(\frac{1}{2}-m)_r}{r!}$;

$A_s=A_s^0=A_{s,0}$; $A_{-s}^m=\dfrac{(-)^s}{(\frac{1}{2})_s\,(m-s)!}$; $q\geqq p$ and p and q are positive integers or zero.

From the definition

$$Q_n(\cos\theta)=\tfrac{1}{2}\{Q_n(\cos\theta+0\cdot i)+Q_n(\cos\theta-0\cdot i)\}$$

we see, after putting $\mu=\cos\theta+0\cdot i$ and $\mu=\cos\theta-0\cdot i$ successively in (2·1), adding and dividing by 2, that (2·1) is also valid for $\mu=\cos\theta$, for real values of θ. We multiply both the sides of (2·1) by $Q_n(\mu)$, where $R(n)>0$, and integrate with respect to μ from $\mu=-1$ to $\mu=+1$, remembering the results

[3] $$(2\cdot 2)\quad \int_{-1}^{1} Q_m(\mu)\,Q_n(\mu)\,d\mu$$

$$=\frac{[\{\psi(n+1)-\psi(m+1)\}\{1+\cos m\pi\cos n\pi\}-\frac{\pi}{2}\sin(n-m)\pi]}{(m-n)(m+n+1)}$$

$=B_{m,n}$ say, where $R(m)>0$, $R(n)>0$ and $m\neq n$, and

[4] $$(2\cdot 3)\quad \int_{-1}^{1}\{Q_n(\mu)\}^2\,d\mu=\frac{1}{2n+1}\left[\frac{\pi^2}{2}-(1+\cos^2 n\pi)\,\psi^{(2)}(n+1)\right]=B_{n,n}\ \text{say},$$

where $R(n)>0$, $\psi(t+1)=\dfrac{d}{dt}\{\log\Gamma(t+1)\}$ and $\psi^{(2)}(z)=\dfrac{d^2}{dz^2}\{\log\Gamma(z)\}$.

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[1] *L.c.*, formula (5·4).

[2] I have to thank a referee for suggesting this notation and other modifications in this paper.

[3] Ganesh Prasad, "On non-orthogonal systems of Legendre's functions," *Proc. Benares Math. Soc.*, 12 (1930), 33-42.

[4] Shabde, "On some series and integrals involving Legendre functions," *Bull. Calcutta Math. Soc.*, 25 (1933), 29.

We have

$$(2\cdot4)\ \int_{-1}^{1} P_p(\mu)\,Q_q(\mu)\,Q_n(\mu)\,d\mu = \sum_{r=0}^{p} \frac{A_r\,A_{p-r}\,A_{q-p+r}}{A_{q+r}}\ \frac{2q-2p+4r+1}{2q-2p+2r+1}\,B_{q-p+2r,\,n}$$

where $R(n) > 0$, $q \geqq p$ and p and q are positive integers or zero.

As a simple deduction from (2·4) we have

$$(2\cdot5)\ \int_{-1}^{1} P_p(\mu)\,Q_q(\mu)\,Q_n(\mu)\,d\mu = 0, \text{ if } q-p+n \text{ is an odd integer.}$$

To evaluate $\int_{0}^{1} P_p(\mu)\,Q_q(\mu)\,Q_n(\mu)\,d\mu$,

where p and q are positive integers or zero such that $q \geqq p$ and $R(n) > 0$, we proceed in the same manner as for (2·4) using in place of (2·2) and (2·3) the results

$$[1]\ (2\cdot6)\ \int_{0}^{1} Q_m\,Q_n\,d\mu$$

$$= \frac{1}{(m-n)(m+n+1)}\left[\psi(n+1)-\psi(m+1)-\pi\cos\frac{m\pi}{2}\sin\frac{n\pi}{2}\cdot\frac{\Pi\left(\frac{n-1}{2}\right)\Pi\left(\frac{m}{2}\right)}{\Pi\left(\frac{m-1}{2}\right)\Pi\left(\frac{n}{2}\right)}\right.$$

$$\left. +\,\pi\cos\frac{n\pi}{2}\sin\frac{m\pi}{2}\,\frac{\Pi\left(\frac{m-1}{2}\right)\Pi\left(\frac{n}{2}\right)}{\Pi\left(\frac{n-1}{2}\right)\Pi\left(\frac{m}{2}\right)}\right]$$

$= C_{m,n}$ say, where $m \neq n$, $R(m) > 0$ and $R(n) > 0$; and

$$(2\cdot7)\ \int_{0}^{1} [Q_n(\mu)]^2\,d\mu = \tfrac{1}{2}\int_{-1}^{1}[Q(\mu)]^2\,d\mu = \frac{1}{2n+1}\left[\frac{\pi}{4} - \psi^{(2)}(n+1)\right]$$

$= C_{n,n}$ say, where n is a positive integer.

We have, finally,

$$(2\cdot8)\ \int_{0}^{1} P_p(\mu)\,Q_q(\mu)\,Q_n(\mu)\,d\mu = \sum_{r=0}^{p} \frac{A_r\,A_{p-r}\,A_{q-p+r}}{A_{q+r}}\ \frac{2q-2p+4r+1}{2q-2p+2r+1}\,C_{q-p+2r,\,n}$$

where $R(n) > 0$, $q \geqq p$, and p and q are positive integers or zero.

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[1] Dhar and Shabde, "On the non-orthogonality of Legendre's functions," *Bull. Calcutta Math. Soc.*, 24 (1932), 185.

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When n is a positive integer (2·4) can be deduced from (2·8), since

$$\int_{-1}^{1} P_p(\mu)\, Q_q(\mu)\, Q_n(\mu)\, d\mu = \{1 + (-1)^{p+q+n}\} \int_0^1 P_p(\mu)\, Q_q(\mu)\, Q_n(\mu)\, d\mu.$$

To evaluate $\int_1^\infty P_p(\mu)\, Q_q(\mu)\, Q_n(\mu)\, d\mu$, where $R(n) > 0$, $q \geqq p$, p and q being positive integers or zero, we use (2·1) with the results

[1] (i) $\int_1^\infty Q_n(\mu)\, Q_m(\mu)\, d\mu = \dfrac{\psi(m+1) - \psi(n+1)}{(m-n)(m+n+1)} = D_{m,n}$ say, where $R(m) > 0$, $R(n) > 0$, $m \neq n$, and

(ii) $\int_1^\infty [Q_n(\mu)]^2\, d\mu = \dfrac{1}{2n+1}\, \psi^{(2)}(n+1) = D_{n,n}$ say, n being a positive integer.

So we have

$$(2·9) \quad \int_1^\infty P_p(\mu)\, Q_q(\mu)\, Q_n(\mu)\, d\mu$$
$$= \sum_{r=0}^{p} \frac{A_r\, A_{p-r}\, A_{q-p+r}}{A_{q+r}}\, \frac{2q-2p+4r+1}{2q-2p+2r+1}\, D_{q-p+2r,\,n}$$

where $R(n) > 0$, $q \geqq p$, q and p being positive integers or zero.

3. Taking Ferrers' definitions for $P_n^m(\mu)$ and $Q_n^m(\mu)$, it can be directly shown that

$$(3·1) \quad \int_{-1}^{1} Q_n^m(\mu)\, P_p^m(\mu)\, d\mu$$
$$= \frac{1}{(p-n)(p+n+1)} \left[\left\{ P_p^m(\mu) \frac{d}{d\mu} Q_n^m(\mu) - Q_n^m(\mu) \frac{d}{d\mu} P_p^m(\mu) \right\} (1-\mu^2) \right]_{-1}^{+1}$$
$$= 0, \text{ if } p+n \text{ is even},$$
$$= \frac{2}{(p-n)(p+n+1)}\, \frac{(p+m)!}{(p-m)!}, \text{ if } p+n \text{ is odd, } p \geqq m.$$

Converting the expansion[2] for $(\mu^2-1)^{\frac{1}{2}m} P_p^m(\mu)\, P_q^m(\mu)$ into Ferrers' notations we get

$$(3·2) \quad (1-\mu^2)^{\frac{1}{2}m} P_p^m(\mu)\, P_q^m(\mu)$$
$$= (-1)^m \frac{(p+m)!\,(q+m)!}{2^m\,(p-m)!\,(q-m)!} \sum_{r=0}^{q+m} \frac{A_{r,m}\, A_{q-r}^m\, A_{p-r}^m}{A_{p+q+m-r}^{-m}} \cdot$$
$$\cdot \frac{2q+2p+2m-4r+1}{2q+2p+2m-2r+1}\, P_{p+q+m-2r}^m(\mu).$$

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[1] Ganesh Prasad, *l.c.*, p. 40.

[2] W. N. Bailey, *l.c.*, formula (3·3).

Multiplying the two sides of (3·2) by $Q_n^m(\mu)$, and integrating with respect to μ from $\mu = -1$ to $\mu = 1$, we have, after using (3·1),

$$(3\cdot 3) \int_{-1}^{1} (1-\mu^2)^{\frac{1}{2}m} P_p^m(\mu)\, P_q^m(\mu)\, Q_n^m(\mu)\, d\mu = 0, \text{ if } p+q+m+n \text{ is even,}$$

and

$$= (-1)^m \frac{(p+m)!\,(q+m)!}{2^m\,(p-m)!\,(q-m)!} \sum_{r=0}^{q+m} \frac{A_{r,m}\, A_{q-r}^m\, A_{p-r}^m}{A_{p+q+m-r}^{-m}} \frac{2q+2p+2m-4r+1}{2p+2q+2m-2r+1} \cdot$$

$$\cdot \frac{2}{(p+q+m-2r-n)} \frac{(p+q+2m-2r)!}{(p+q+m-2r+n+1)\,(p+q-2r)!}$$

if $p+q+m+n$ is odd such that $p+q>0$, $p-q \geqq 2m$ and $n < p-q-m$ or $> p+q+m$

4. To evaluate $\int_{-1}^{1} P_p^m(\mu)\, P_q^m(\mu)\, P_n^t(\mu)\,(1-\mu^2)^{\frac{1}{2}m}\, d\mu$, we use (3·2) with the result[1]

$$(4\cdot 1) \int_{-1}^{1} P_q^m(x)\, P_n^t(x)\, dx = 0 \text{ if } q+n \text{ is odd,}$$

$$= (-1)^k \frac{2}{2n+1} \frac{(n+t)!}{(n-m)!}, \text{ if } n=q \text{ and } m-t=2k,$$

$$= \frac{4\,(n+t)!}{(n-t)!} \Big[\sum_{s=0}^{r,k} (-1)^{k+s}\, {}^kC_s\, {}^{r+k-s-1}C_{k-1}\, (2n+2k-4s+1)$$

$$\frac{(q-t-2s)!\,(m+q)!\,(2q-2s)!\,(q-s+k)!\,(q-r-s)!}{(q+m-2s)!\,(q-m)!\,(q-s)!\,(q-r-s+k-1)!}$$

$$\frac{(2q-2k-2r-2s-1)!}{(2q-2r-2s+1)!\,(2q+2k-2s+1)!} \Big] \text{ where } n=q-2r,$$

$m-t=2k$, and the summation is taken over all values of s from 0 to r if $r<k$, and all values of s from 0 to k if $r>k$.

Hence if $p \geqq m$, $q \geqq m$, $n \geqq t$, $p-q \geqq 2m$ and $m-t=2k$, k being any positive integer, we have

$$(4\cdot 2) \int_{-1}^{1} (1-\mu^2)^{\frac{1}{2}m} P_p^m(\mu)\, P_q^m(\mu)\, P_n^t(\mu)\, d\mu = 0 \text{ when } p+q+m+n \text{ is}$$

an odd integer, and

$$= (-1)^m \frac{(p+m)!\,(q+m)!}{2^m\,(p-m)!\,(q-m)!} \sum_{l=0}^{q+m} \frac{A_{l,m}\, A_{q-l}^m\, A_{p-l}^m}{A_{p+q+m-l}^{-m}} \frac{2p+2q+2m-4l+1}{2p+2q+2m-2l+1} \cdot$$

$$\cdot \int_{-1}^{1} P_n^t(\mu)\, P_{p+q+m-2l}^m(\mu)\, d\mu,$$

[1] H. K. Sirkar, *Proc. Edin. Math. Soc.*, 2, 1 (1927-29), 244, "On the evaluation of $\int_{-1}^{1} P_n^m P_q^p\, dx$ and $\int_0^1 P_n^m P_q^p\, dx$."

where $m \leqq n < p - q - m < p + q + m$,

$$= \frac{(-1)^m (p+m)!\,(q+m)!}{2^m (p-m)!\,(q-m)!} \sum_{l=0}^{q+m} \frac{A_{l,m}\, A_{q-l}^m\, A_{p-l}^m}{A_{p+q+m-l}^{-m}} \, \frac{2p+2q+2m-4l+1}{2p+2q+2m-2l+1}$$

$$\frac{4\,(n+t)!}{(n-t)!} \Bigg[\sum_{s=0}^{r,k} (-1)^{k+s}\, {}^{k}C_s \, {}^{r+k-s-1}C_{k-1}\, (2n+2k-4s+1)! \, \frac{(p+q+m-2l-t-2s)!\,(p+q+2m-2l)!\,(2p+2q+2m-4l-2s)!}{(p+q+2m-2l-2s)!\,(p+q-2l)!\,(p+q+m-2l-s)!}$$

$$\frac{(p+q+m-2l-s+k)!\,(p+q+m-2l-r-s)!\,(2p+2q+2m-4l+2k-2r-2s-1)!}{(p+q+m-2l-r-s+k-1)!\,(2p+2q+2m-4l-2r-2s+1)!\,(2p+2q+2m-4l+2k-2s+1)!} \Bigg]$$

where $p + q + m - 2l - 2r = n$, the summation within [ ] being taken over all values of s from 0 to r if $r < k$, and all values of s from 0 to k if $r > k$.

5. Following the method given by Ganesh Prasad[1] for evaluating $\int_{-1}^{1} Q_m(\mu)\, Q_n(\mu)\, d\mu$ for *unrestricted* values of m and n such that $R(m) > 0$, and $R(n) > 0$, the following results can be easily worked out:—

(5·1) $\int_{-1}^{1} P_n(\mu)\, Q_m(\mu)\, d\mu$

$$= \frac{-1 + \cos(m-n)\pi - \dfrac{2}{\pi} \sin n\pi \cos m\pi \,\{\psi(m+1) - \psi(n+1)\}}{(m-n)(m+n+1)},$$

where $R(m) > 0$, $R(n) > 0$ and $m \neq n$.

(5·2) $\int_{-1}^{1} Q_n(\mu)\, d\mu = -\dfrac{1 - \cos n\pi}{n(n+1)}$, where $R(n) > 0$,

(5·3) $\int_{0}^{1} P_n(\mu)\, Q_m(\mu)\, d\mu$

$$= \frac{1}{(m-n)(m+n+1)} \left[-1 + \frac{\Pi\left(\frac{n}{2}\right) \Pi\left(\frac{m-1}{2}\right)}{\Pi\left(\frac{n-1}{2}\right) \Pi\left(\frac{m}{2}\right)} \cos\left(\frac{n-m}{2}\right)\pi \right],$$

where $R(m) > 0$, $R(n) > 0$ and $m \neq n$.

These results reduce to the values given by Nicholson[2], when m and n are positive integers.

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[1] *L.c.*, p. 38.

[2] *Phil. Mag.* (6), 43 (1922), 1-29.

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Paper II

On some definite Integrals involving Legendre Functions.

By N. G. SHABDE, *Research Student, Mathematical Department, Edinburgh University.*

Introduction.

IN a recent paper †, the author has obtained some definite integrals involving Legendre functions. The object of the present note is to generalize some of the results given in this paper. Also use is made of a theorem of MacRobert ‡ to evaluate an integral, the integration being with respect to the degree of the Legendre functions. Such integrals with respect to the degrees of the Legendre functions have been a subject of study in a recent paper § by MacRobert. As little is known regarding such integrals, it will be worth while studying such integrals.

† Proc. Edin. Math. Soc. ser. 2, iv. part i. pp. 41–46. This paper will be referred to as (P. 1) in this note. I take this opportunity of stating that (3.3) of (P. 1) is also valid if $n>p+q+m$.

‡ Proc. Roy. Soc. Edin. li. part ii. (no. 16), pp. 116–126. This paper will be referred to as (P. 2). The theorem is given on p. 123.

§ Proc. Roy. Soc. Edin. liv. part ii. (no. 13), pp. 135–144.

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§ 1.

If p and q are positive integers or zero, $R(n)>0$, such that $p+q>0$, $p \geq q$, and $R(n)<p-q$ or $>p+q$, then we have

$$\int_{-1}^{1} P_p(\mu) . P_q(\mu) . Q_n(\mu)\, d\mu$$

$$= \sum_{r=0}^{q} \frac{A_{p-r} . A_r . A_{q-r}}{A_{p+q-r}} \left(\frac{2p+2q-4r+1}{2p+2q-2r+1}\right) \times \int_{-1}^{1} P_{p+q-2r}(\mu) . Q_n(\mu)\, d\mu$$

$$= \sum_{r=0}^{q} \frac{A_{p-r} . A_r . A_{q-r}}{A_{p+q-r}} \left(\frac{2p+2q-4r+1}{2p+2q-2r+1}\right) \times \left[\frac{-1+\cos(n-p-q+2r)\pi}{(n-p-q+2r)(n+p+q-2r+1)}\right] \quad . \quad (1.1)$$

by means (5.1) of (P. 1). Here $A_r = \dfrac{2^r (\frac{1}{2})_r}{r\,!}$, as in (P. 1).

Similarly, with the same restrictions on p, q, and n as in (1.1), we have

$$\int_{0}^{1} P_p(\mu) . P_q(\mu) . Q_n(\mu)\, d\mu$$

$$= \sum_{r=0}^{q} \frac{A_{p-r} . A_r . A_{q-r}}{A_{p+q-r}} \left(\frac{2p+2q-4r+1}{2p+2q-2r+1}\right) \int_{0}^{1} P_{p+q-2r}(\mu) . Q_n(\mu)\, d\mu$$

$$= \sum_{r=0}^{q} \frac{A_{p-r} . A_r . A_{q-r}}{A_{p+q-r}} \left(\frac{2p+2q-4r+1}{2p+2q-2r+1}\right) \times \frac{1}{(n-p-q+2r)(n+p+q-2r+1)} \times \left[-1 + \frac{\Pi\left(\frac{p+q-2r}{2}\right) \Pi\left(\frac{n-1}{2}\right)}{\Pi\left(\frac{p+q-2r-1}{2}\right) \Pi\left(\frac{n}{2}\right)} \cos\left(\frac{p+q-2r-n}{2}\right)\pi\right] \quad . \quad . \quad . \quad (1.2)$$

by means of (5.3) of (P. 1).

§ 2.

MacRobert's Fourier-Legendre integral theorem in (P. 2) states that

$$\text{“if } F(\lambda)=\int_p^q f(\phi)\,.\,P_{\lambda-\frac{1}{2}}(\cos\phi)\sqrt{\sin\phi}\,d\phi,\ 0\leq p<q\leq\pi,$$

then

$$2\int_0^\infty F(\lambda)\,.\,\lambda\,.\,\sin(\pi\lambda)\,.\,P_{\lambda-\frac{1}{2}}(-\cos\theta)\sqrt{\sin\theta}\,.\,d\lambda$$

$$=\left\{\begin{array}{ll}\frac{1}{2}\{f(\theta+0)+f(\theta-0)\}. & p<\theta<q\\ 0 & ,\ 0<\theta<p \text{ or } q<\theta<\pi\end{array}\right\}.\text{”}$$

We use this theorem with the integral

$$\int_{-1}^1 P_m(\mu)\,.\,P_n(\mu)\,d\mu=\frac{-1}{(m-n)(m+n+1)}$$

$$\times\left[\frac{4}{\pi^2}\sin m\pi\,.\,\sin n\pi\{\psi(m+1)-\psi(n+1)\}+\frac{2}{\pi}\sin(n-m)\pi\right]. \quad . \quad . \quad . \quad (2.1)\,*$$

This result is valid if $m\neq n$ and $m+n+1\neq 0$. Putting $n=\lambda-\frac{1}{2}$ and taking m such that $-\frac{1}{2}<m<0$, we have

$$\int_{-1}^1 P_m(\mu)\,.\,P_{\lambda-\frac{1}{2}}(\mu)\,d\mu=\frac{+1}{(m-\lambda+\frac{1}{2})(m+\lambda+\frac{1}{2})}$$

$$\times\left[\frac{4}{\pi^2}\sin m\pi\cos\lambda\pi\{\psi(m+1)-\psi(\lambda+\tfrac{1}{2})\}+\frac{2}{\pi}\cos(\lambda-m)\pi\right]. \quad . \quad . \quad . \quad (2.2)$$

Using the theorem stated above, we get

$$\int_0^\infty \lambda\sin\lambda\pi\,.\,P_{\lambda-\frac{1}{2}}(-\cos\theta)$$

$$\times\left[\frac{2}{(m-\lambda+\frac{1}{2})(m+\lambda+\frac{1}{2})}\left\{\frac{4}{\pi^2}\sin m\pi\cos\lambda\pi\right.\right.$$

$$\left.\left.\times\{\psi(m+1)-\psi(\lambda+\tfrac{1}{2})\}+\frac{2}{\pi}\cos(\lambda-m)\pi\right\}d\lambda\right]$$

$$=P_m(\cos\theta);\quad -\frac{1}{2}<m<0 \text{ and } 0<\theta<\pi. \quad . \quad (2.3)$$

* Ganesh Prasad, "On Non-orthogonal System of Legendre Functions," Proc. Benares Math. Soc. xii. pp. 33–42 (1930).

Paper III

ON THE VALUE OF $\int_{-1}^{1} Q_m(\mu)\, Q_n(\mu)\, d\mu.$

BY

N. G. SHABDE

1. The object of the present note is (1) to examine carefully the integral $\int_{-1}^{1} Q_m(\mu).\, Q_n(\mu).\, d\mu$ for values of m and n, which are integers and different, and (2) to show that the result of Nicholson* that the integral is equal to zero, whatever different integers m and n may be, is wrong. The correct result has been given by Ganesh Prasad, in his paper,† "On non-orthogonal systems of Legendre's functions."

Ganesh Prasad has proved that‡ "if m and n are integers but both odd or both even, then the system is non-orthogonal," that is, in this case of m and n being both odd or both even integers, $\int_{-1}^{1} Q_m.\, Q_n.\, d\mu \neq 0$ when $m \neq n$. Thus, there is a difference in the value of $\int_{-1}^{1} Q_m\, Q_n\, d\mu$ as given by Ganesh Prasad and as that given by Nicholson for the case of m and n being both odd or both even integers and $m \neq n$, according to Ganesh Prasad the value being *not equal to zero* and according to Nicholson it being *equal to zero*. For other integral values of

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* See in the *Phil. Mag*, (6) Vol XLIII, pp, 1—29, (1922), the paper by Prof. J. W. Nicholson, F. R. S, entitled "Zonal Harmonics of the second kind."

† The *Proc. Benares Math. Society*, Vol XII, 1930, pp, 33—42.

‡ Page 39, IIIrd deduction.

2 The Value Of $\int_{-1}^{1} Q_m(\mu)\, Q_n(\mu) d\mu.$

m and n, $m \neq n$, the value of the integral $\int_{-1}^{1} Q_m\ Q_n\, d\mu = 0$ according to both.

2. Let us first state the proof of Prof. Nicholson*:—

$Q_m(\mu)$ may be defined as

$$\tfrac{1}{2}P_m(\mu) \log \frac{1+\mu}{1-\mu} - \sum_{r=0} a_r\ P_{m-2r-1},\ \mu < 1,$$

and the coefficients a_r are numerical.

$$\int_{-1}^{1} Q_m\, Q_n\, d\mu = \frac{1}{(m-n)(m+n+1)} \Big[(1-\mu^2)\ (Q_m Q'_n - Q_n Q'_m) \Big]_{-1}^{1}.$$

We shall suppose that μ lies between ± 1 and m and n are integers.

$$\begin{aligned}
&(1-\mu^2)\ (Q_m Q'_n - Q_n\ Q'_m) \\
&= (1-\mu^2) \sum_{r=0} a_r\ P_{m-2r-1} \sum_{s=0} a_s\ P'_{n-2s-1} \\
&\quad - (1-\mu^2)\ \Sigma a_r\ P'_{m-2r-1}\ \Sigma a_s\ P_{n-2s-1} \\
&\quad - \tfrac{1}{2}\ (1-\mu^2)\ P'_n \log \frac{1+\mu}{1-\mu}\ \Sigma a_r\ P_{m-2r-1} \\
&\quad + \tfrac{1}{2}\ (1-\mu^2)\ P'_m \log \frac{1+\mu}{1-\mu}\ \Sigma a_s\ P_{n-2s-1}.
\end{aligned}$$

P and P′ are polynomials and finite for all values of μ concerned. Therefore, the first two rows vanish by virtue of the factor $1-\mu^2$. The next two rows also vanish because the limit of $(1-\mu^2) \log \frac{1+\mu}{1-\mu}$ is zero when $\mu = \pm 1$. Therefore $\int_{-1}^{1} Q_m\ Q_n\ d\mu = 0.$

3. *We shall now examine this proof critically.*

Let us find the value of

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* *L. c.*, pp 1–2.

$(1-\mu^2)\,(Q_m.Q'_n - Q_n.\,Q'_m)$ and then see whether it agrees with that given by Nicholson.

$$Q'_n(\mu) = \frac{dQ_n}{d\mu} = \tfrac{1}{2}\,P'_n \log \frac{1+\mu}{1-\mu} + P_n \frac{1}{1-\mu^2} - \sum_{s=0} a_s.\,P'_{n-2s-1},$$

$$Q'_m(\mu) = \frac{dQ_m}{d\mu} = \tfrac{1}{2}\,P'_m. \log \frac{1+\mu}{1-\mu} + P_m.\,\frac{1}{1-\mu^2} - \sum_{r=0} a_r.P'_{m-2r-1},$$

$$Q_n(\mu) = \tfrac{1}{2}P_n(\mu).\log \frac{1+\mu}{1-\mu} - \sum_{s=0} a_s\,P_{n-2s-1},$$

$$Q_m(\mu) = \tfrac{1}{2}P_m(\mu).\log \frac{1+\mu}{1-\mu} - \sum_{r=0} a_r\,P_{m-2r-1}.$$

Hence

$$(1-\mu^2)\left(Q_m.\,\frac{dQ_n}{d\mu} - Q_n.\,\frac{dQ_m}{d\mu}\right)$$

$$=(1-\mu^2)\,\Bigg[\left\{\tfrac{1}{2}P_m(\mu).\log\frac{1+\mu}{1-\mu} - \sum_{r=0} a_r.\,P_{m-2r-1}\right\}$$

$$\times\left\{\tfrac{1}{2}P'_n.\log\frac{1+\mu}{1-\mu} + P_n.\,\frac{1}{1-\mu^2} - \sum_{s=0} a_s.\,P'_{n-2s-1}\right\}$$

$$-\left\{\tfrac{1}{2}P_n.\log\frac{1+\mu}{1-\mu} - \sum_{s=0} a_s.\,P_{n-2s-1}\right\}\times$$

$$\times\left\{\tfrac{1}{2}P'_m.\log\frac{1+\mu}{1-\mu} + P_m.\,\frac{1}{1-\mu^2} - \sum_{r=0} a_r\,P'_{m-2r-1}\right\}\Bigg]$$

= The rows given in Nicholson's proof

$$+(1-\mu^2).\,\tfrac{1}{4}.\,P_m.\,P'_n.\left(\log\frac{1+\mu}{1-\mu}\right)^2$$

$$-(1-\mu^2).\,\tfrac{1}{4}.\,P'_m.P_n.\left(\log\frac{1+\mu}{1-\mu}\right)^2$$

$$+(1-\mu^2)\left\{\tfrac{1}{2}\,P_m.\,P_n.\,\frac{1}{1-\mu^2}\log\frac{1+\mu}{1-\mu}\right.$$

$$\left.-\tfrac{1}{2}\,\frac{1}{1-\mu^2}\,P_m.\,P_n.\log\frac{1+\mu}{1-\mu}\right\}$$

$$+(1-\mu^2)\left\{\frac{P_m}{1-\mu^2}.\sum_{s=0} a_s.P_{n-2s-1} - \frac{P_n}{1-\mu^2}.\sum_{r=0} a_r.\,P_{m-2r-1}\right\} + T,$$

The Value Of $\int_{-1}^{1} Q_m(\mu)\, Q_n(\mu) d\mu.$

where

$$T \text{ is } \tfrac{1}{2}(1-\mu^2)\log\frac{1+\mu}{1-\mu}\left\{P_n\,\Sigma a_r\,P'_{m-2r-1} - P_m\,\Sigma a_s\,P'_{n-2s-1}\right\}$$

$=$ The rows given by Nicholson $+$

$$T + (1-\mu^2).\tfrac{1}{4}.\,P_m.P'_n.\left(\log\frac{1+\mu}{1-\mu}\right)^2 - \frac{1-\mu^2}{4}.\,P'_m.\,P_n\left(\log\frac{1+\mu}{1-\mu}\right)^2$$
$$+\,P_m\,\{\Sigma a_s.\,P_{n-2s-1}\} - \{P_n.\,\Sigma a_r.\,P_{m-2r-1}\}.$$

So our expression for $(1-\mu^2)\,(Q_m.\,Q'_n - Q_n.\,Q'_m)$ differs from that given by Nicholson by the last two rows. When we substitute the limits ± 1 for μ, the rows of Nicholson vanish and also the first of our rows vanishes as the limit of $(1-\mu^2)$. $\left(\log\frac{1+\mu}{1-\mu}\right)^2$ is zero when $\mu=\pm 1$ as will be seen by evaluating the undetermined form.

*Now coming to our last row we see that it *cannot vanish* if the limits are substituted, when m and n are *both odd or both even* integers.

Thus, it is definitely seen that

$$\int_{-1}^{1} Q_m\, Q_n\, d\mu \neq 0 \quad \text{when } m \neq n \text{ and } m \text{ and } n \text{ are both odd or both even integers.}$$

Thus, it is clear that the result given by Ganesh Prasad that "if m and n are integers but both odd or both even, then the system (of Legendre's functions Q_n of the second kind) is non-orthogonal" is correct,

In conclusion, I wish to express my best thanks to Prof. Ganesh Prasad for the kind interest he has taken in the preparation of this note.

* See next page.

*

$$\left[P_m(\mu) \sum a_s P_{n-2s-1}(\mu) - P_n(\mu) \sum a_r P_{m-2r-1}(\mu) \right]_{-1}^{+1}$$

$$= \left\{ 1 + (-1)^{m+n} \right\} \left[\sum_{s=0}^{s=\left[\frac{n-1}{2}\right]} a_s - \sum_{r=0}^{r=\left[\frac{m-1}{2}\right]} a_r \right]$$

$$= 2 \cos^2 (m+n) \frac{\pi}{2} \sum_{r=m+1}^{n} \frac{1}{r}$$

This cannot vanish when m and n are both odd or both even integers.

4. Let us work out a special case by actual calculations. If we use the expansions

$$Q_0 = \sum_{n=0}^{\infty} \left(\frac{1}{2n+1} + \frac{1}{2n+2} \right) P_{2n+1}$$

$$Q_1 = \sum_{n=0}^{\infty} \left(\frac{1}{2n-1} + \frac{1}{2n+2} \right) P_{2n}$$

$$Q_2 = \sum_{n=0}^{\infty} \left(\frac{1}{2n-1} + \frac{1}{2n+4} \right) P_{2n+1}$$

$$Q_3 = \sum_{n=0}^{\infty} \left(\frac{1}{2n-3} + \frac{1}{2n+4} \right) P_{2n}$$

The value of the integral may be calculated by means of the Parseval's theorem for Legendre series. In particular

$$\frac{1}{2}\int_{-1}^{1} Q_0\, Q_2\, d\mu = \frac{1}{3}\left(\frac{1}{1}+\frac{1}{2}\right)\left(-\frac{1}{1}+\frac{1}{4}\right) + \frac{1}{7}\left(\frac{1}{3}+\frac{1}{4}\right)\left(\frac{1}{1}+\frac{1}{9}\right)$$
$$+ \frac{1}{11}\left(\frac{1}{5}+\frac{1}{6}\right)\left(\frac{1}{3}+\frac{1}{11}\right) + \frac{1}{15}\left(\frac{1}{7}+\frac{1}{8}\right)\left(\frac{1}{5}+\frac{1}{13}\right) + \cdots$$

To prove that the integral is not zero we have to prove that

$$\frac{1}{2} \neq \frac{1}{1\cdot 2}\cdot\frac{1}{4} + \frac{1}{3\cdot 4}\left(\frac{1}{1}+\frac{1}{9}\right) + \frac{1}{5\cdot 6}\left(\frac{1}{3}+\frac{1}{11}\right) + \cdots$$

That is, that

$$\frac{3}{8} \neq \sum_{n=0}^{\infty} \frac{1}{(2n+3)(2n+4)} \left[\frac{1}{2n+1} + \frac{1}{2n+9} \right]$$

Now

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+3)(2n+4)} < \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)(2n+3)}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+3)(2n+4)(2n+9)} < \sum_{n=0}^{\infty} \frac{1}{(2n+3)(2n+4)(2n+5)}$$

Hence the series is less than

$$\left\{\left(\frac{1}{1\cdot 2\cdot 3}\right)+\left(\frac{1}{3\cdot 4\cdot 5}\right)+\left(\frac{1}{5\cdot 6\cdot 7}\right)+\cdots\right\}$$

$$+\left(\frac{1}{3\cdot 4\cdot 5}+\frac{1}{5\cdot 6\cdot 7}+\cdots\right)=2\left(\frac{1}{1\cdot 2\cdot 3}+\frac{1}{3\cdot 4\cdot 5}+\cdots\right)-\frac{1}{6}$$

$$=\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}-\frac{1}{4\cdot 5}+\cdots-\frac{1}{6}$$

$$=1-\frac{1}{2}-\frac{1}{2}+\frac{1}{3}+\cdots-\frac{1}{6}$$

$$=2\log 2-1-\frac{1}{6}=\cdot 32$$ approximately.

This is certainly less than $\frac{3}{8}$.

Paper IV

On the Summation of Infinite Series of Legendre's Polynomials.

By

N. G. Sharde.

(Calcutta University)

The object of the present paper is to add to the list of such series of Legendre's polynomials of the type $\sum_{n=0}^{\infty} a_n P_n(x)$, a_n being independent of the argument x, as admit of being summed up into forms, compact and *free from the sign of integration.*

Among those who have given such series may be mentioned Bauer,* Heine,† Most,‡ Routh§, Hargreaves,¶ Chapman,‖ Darling,** Stuart and Ganesh Prasad.††

Many of the series given below are believed to be new and these are starred. There are also numbers of other series, which deserve special attention, as, although they can be summed up by combining certain known and new series or by known methods used by previous writers, their sums are so simple that they merit to be added to the list. These series are not starred.

* "Von den Coefficienten der Reihen von Kugelfunctionen einer Variablen" *Crelle's Journal*, Bd. 56. pp. 101-121.

† *Handbuch der Kugelfunktionen*, Bd I.

‡ "Ueber die Differentialquotienten der Kugelfunctionen" *Crelle's Journal*, Bd. 70, pp. 163-168.

§ *Proceedings London Mathematical Society*, Vol. XXVI, pp. 481-491.

¶ "Expansion of Elliptic integrals by zonal harmonics with some derived integrals and series." *The Messenger of Mathematics*, Vol. XXVI, 1897, pp. 89-98.

‖ "On the expansion of $(1-2r\cos\theta+r^2)^n$ in a series of Legendre's functions." *Quarterly Journal of Mathematics*, Vol. XLVII, 1916, pp. 16-26.

** "On Legendre's coefficients and Associated functions with non-integral subscripts and their connection with the elliptic integrals." *Q. J. of Mathematics*, Vol. XLIX, pp. 289-301.

†† "On the summation of infinite series of Legendre's functions," *Bulletin of the Calcutta Mathematical Society*, Vol. XXII, pp. 159 170.

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My sincere thanks are due to Professor Ganesh Parsad, at whose kind suggestion I undertook this problem of summation.

In the first part of this paper is given the list of the series along with their sums and more or less full hints for obtaining the results are given in the second part of the paper.

Unless otherwise stated, the argument of the Legendre Polynomial P_n is understood to be $x = \cos\theta$ and $|a|$ or $|r|$, which occurs in the series, is supposed to be less than unity.

PART I.

(1) $$\sum_{n=0}^{\infty} \frac{r^{3(n+1)}}{3(n+1)} P_n = \tfrac{1}{3}\log\left[\frac{r^3 - x + \sqrt{1-2xr^3+r^6}}{1-x}\right]$$

(2) $$\sum_{n=0}^{\infty} \frac{P_{2n+1}}{2n+1} = \tfrac{1}{2}\log\left[\cot\frac{\theta}{2}\cdot\frac{1+\cos\frac{\theta}{2}}{1+\sin\frac{\theta}{2}}\right].$$

(3) $$\sum_{n=0}^{\infty} \frac{P_n}{(n+1)(n+2)(n+3)}$$

$$= \left[\left\{\frac{3x^2-4x+1}{4}\right\}\left\{\log\frac{1-x+\sqrt{2}\,\sqrt{1-x}}{1-x}\right\}\right.$$

$$\left. - \frac{3}{2\sqrt{2}}\,(1-x)\sqrt{1-x} - \frac{3}{4}\,x + 1\right]$$

(4) $$\sum_{n=0}^{\infty} \frac{(-1)^n P_n}{(n+1)(n+2)(n+3)}$$

$$= \left[\left\{\frac{3x^2+4x+1}{4}\right\}\left\{\log\frac{1+x+\sqrt{2}\sqrt{1+x}}{1+x}\right\}\right.$$

$$\left. - \frac{3}{2\sqrt{2}}\,(1+x)\sqrt{1+x} + \frac{3}{4}\,x + 1\right]$$

$$(5)\quad \sum_{n=0}^{\infty} \frac{P_n}{(n+1)(n+2)(n+3)(n+4)}$$

$$= \frac{1}{6}\left[\left\{\frac{-5x^3+9x^2-3x-1}{2}\right\} \log \frac{1-x+\sqrt{2(1-x)}}{1-x}\right.$$

$$-\left\{\frac{15x^2-22x+7}{6}\right\}\sqrt{2(1-x)}$$

$$\left.+\left\{\frac{15x^2-27x+14}{6}\right\}\right].$$

$$6)\quad \sum_{n=0}^{\infty} \frac{(-1)^n P_n}{(n+1)(n+2)(n+3)(n+4)}$$

$$= \frac{1}{6}\left[\left\{\frac{5x^3+9x^2+3x-1}{2}\right\} \log \frac{1+x+\sqrt{2(1+x)}}{1+x}\right.$$

$$-\left\{\frac{15x^2+22x+7}{6}\right\}\sqrt{2(1+x)}$$

$$\left.+\left\{\frac{15x^2+27x+14}{6}\right\}\right].$$

$$(7)\quad \sum_{n=0}^{\infty} \frac{a^{n+5} P_n}{n+5} = \left[\frac{a^3\sqrt{1-2ax+a^2}}{4} + \frac{7}{4\cdot 3}\, x.\, a^2\sqrt{1-2ax+a^2}\right.$$

$$+\frac{a}{2}\sqrt{1-2ax+a^2}\left\{\frac{5}{3}\cdot\frac{7}{4}\, x^2 - \frac{3}{4}\right\}$$

$$+\sqrt{1-2ax+a^2}\left\{-\frac{7\cdot 2}{3\cdot 4}\, x + \frac{3.5}{2.3}\cdot\frac{7}{4}\cdot x^3 - \frac{3.3}{4.2}\, x\right\}$$

$$+\left\{\frac{2.7}{3.4}\, x - \frac{5.3.7}{2.3.4} x^3 + \frac{3.3}{4,2}\, x\right\}$$

$$+\left\{\log \frac{a-x+\sqrt{1-2ax+a^2}}{1-x}\right\}$$

$$\left.\times\left\{\frac{7}{4}\, x\left(\frac{3.5}{2.3}\, x^3 - \frac{1.5}{2.3}\, x - \frac{2}{3}\, x\right) - \frac{3}{4}\left(\frac{3x^2}{2} - \frac{1}{2}\right)\right\}\right]$$

$$(8)\quad \sum_{n=0}^{\infty} \frac{P_n}{n+5} = \Big[\sqrt{2(1-x)} \Big\{ \frac{1}{4} + \frac{7}{4.3}\, x + \frac{1}{2} \cdot \frac{5}{3} \cdot \frac{7}{4}\, x^2 - \frac{3}{2.4}$$

$$- \frac{7.2}{4.3}\, x + \frac{7}{4} \cdot \frac{3}{2} \cdot \frac{5}{3}\, x^3 - \frac{3.3}{4.2}\, x \Big\} + \Big\{ \frac{2}{3} \cdot \frac{7}{4}\, x - \frac{5.3.7}{2.3.4}\, x^3 + \frac{3.3}{4.2}\, x \Big\}$$

$$+ \Big\{ \log \frac{1-x+\sqrt{2(1-x)}}{1-x} \Big\} \Big\{ \frac{7}{4}\, x \Big(\frac{3.5.x^3}{2.3} - \frac{1.5}{2.3}\, x - \frac{2}{3}\, x \Big)$$

$$- \frac{3}{4} \Big(\frac{3x^2}{2} - \tfrac{1}{2} \Big) \Big\} \Big]$$

$$(9)\quad \sum_{n=0}^{\infty} \frac{(-1)^n P_n}{n+5} = \Big[\sqrt{2(1+x)} \Big\{ \frac{1}{4} - \frac{7}{4.3}\, x + \frac{1}{2} \cdot \frac{5}{3} \cdot \frac{7}{4}\, x^2 - \frac{3}{2.4}$$

$$+ \frac{7.2}{4.3}\, x - \frac{7}{4} \cdot \frac{3}{2} \cdot \frac{5}{3}\, x^3 + \frac{3.3}{4.2}\, x \Big\} + \Big\{ - \frac{2}{3} \cdot \frac{7}{4}\, x + \frac{5.3.7}{2.3.4}\, x^3 - \frac{3.3}{4.2}\, x \Big\}$$

$$+ \Big\{ \log \frac{1+x+\sqrt{2(1+x)}}{1+x} \Big\} \Big\{ - \frac{7}{4}\, x \Big(- \frac{3.5.x^3}{2.3}$$

$$+ \frac{1.5}{2.3}\, x + \frac{2}{3}\, x \Big) - \frac{3}{4} \Big(\frac{3x^2}{2} - \tfrac{1}{2} \Big) \Big\} \Big]$$

$$(10)\quad \sum_{n=0}^{\infty} \frac{(-1)^{n-1} P_{2n+1}}{2n+3} = x \tan^{-1} \frac{1}{\sqrt{x}} - \sqrt{x}$$

$$(11)\quad \sum_{n=0}^{\infty} \frac{(-1)^{n-1} P_{2n}}{2(n+1)} = \sqrt{x} + x \log \frac{1-\sqrt{x}}{1-x} + \frac{x}{2} \log (1+x) - 1$$

$$(12)\quad \sum_{n=0}^{\infty} \frac{(-1)^n P_{2n+1}}{2n+4} = \frac{\sqrt{x}}{2} + \frac{3x}{2} (\sqrt{x} - 1)$$

$$+ \Big(\frac{3x^2}{2} - \tfrac{1}{2} \Big) \frac{1}{2} \Big[\log (1+x) \Big] + \Big(\frac{3x^2}{2} - \frac{1}{2} \Big) \log \frac{1-\sqrt{x}}{1-x}$$

$$(13)\quad \sum_{n=0}^{\infty} \frac{(-1)^{n-1} P_{2n}}{2n+3} = \frac{\sqrt{x}}{2} - 3 .\, \frac{x^{\frac{3}{2}}}{2} + \Big(\frac{3x^2}{2} - \tfrac{1}{2} \Big) \tan^{-1} \frac{1}{\sqrt{x}}$$

$$(14)\quad \sum_{n=0}^{\infty} \frac{(-1)^n P_{2n}}{2(n+2)} = \Big[-\frac{\sqrt{x}}{3} + \frac{5}{3}\cdot\frac{x^{\frac{3}{2}}}{2}$$

$$+ \left(\frac{5x^2}{2} - \frac{2}{3} \right) \left\{ \sqrt{x} + \frac{x}{2} \log(1+x) - 1 + x \log \frac{1-\sqrt{x}}{1-x} \right\}$$

$$- \frac{5}{3}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot \log(1+x) - \frac{5}{6} \log \frac{1-\sqrt{x}}{1-x} \Big]$$

$$(15)\quad \sum_{n=0}^{\infty} \frac{(-1)^n P_{2n+1}}{2n+5} = \Big[\frac{\sqrt{x}}{3} + \frac{5}{3.2} x^{\frac{3}{2}}$$

$$+ \left(\frac{5}{2} x^2 - \tfrac{2}{3} \right) \left(-\sqrt{x} + x \tan^{-1} \frac{1}{\sqrt{x}} \right) - \frac{5}{3.2} \tan^{-1} \frac{1}{\sqrt{x}} \Big]$$

$$(16)\quad \sum_{n=0}^{\infty} \frac{P_{4n+1}}{4n+1} = \frac{1}{4} \left[\cot \frac{\theta}{2} \; \frac{1+\cos\frac{\theta}{2}}{1+\sin\frac{\theta}{2}} \right] + \frac{1}{2} \left[\tan^{-1} (\sqrt{\cos\theta}) \right]$$

$$*(17)\quad P_0 + \sum_{n=1}^{\infty} \frac{1.3.5\ldots 2n-1}{2.4\ldots 2n} P_n = \frac{2K}{\pi\{2(1-x)\}^{\frac{1}{4}}},$$

K being the complete elliptic integral of the first kind, with $(\text{modulus})^2 = \frac{1}{2}\left(1 - \sin\frac{\theta}{2}\right)$.

$$*(18)\quad P_0 + \sum_{n=1}^{\infty} (-1)^n \frac{1.3.5\ldots 2n-1}{2.4\ldots 2n} P_n = \frac{2}{\pi} \cdot \frac{K}{\{2(1+x)\}^{\frac{1}{4}}},$$

K having the same meaning as in (17) with $(\text{modulus})^2 = \frac{1}{2}\left(1-\cos\frac{\theta}{2}\right)$

$$*(19)\quad \sum_{n=0}^{\infty} \frac{r^{2n+3}}{2n+3} P_n = \frac{1}{2} \left[u - 2 \,\text{dn}\, u.\ \text{cs}\, u + 2 \,\text{ds}\, u - 2E(u) \right],$$

where $u = \text{cn}^{-1} \left\{ \frac{1-r^2}{1+r^2}, \cos\frac{\theta}{2} \right\}$ and E (u) is the elliptic integral of the second kind, the modulus k of these elliptic functions being $\cos\frac{\theta}{2}$.

$$*(20)\quad \sum_{n=0}^{\infty} \frac{P_n}{2n+3} = \left[\frac{K}{2} - E + \sin\frac{\theta}{2}\right],$$

K and E being complete elliptic integrals of the first and second kind with modulus $k = \cos\frac{\theta}{2}$.

$$*(21)\quad \sum_{n=0}^{\infty} \frac{(-1)^n r^{2n+3}}{2n+3} P_n = \frac{1}{2}\left[u - 2\,\mathrm{dn}\,u\,\mathrm{cs}\,u + 2\,\mathrm{ds}\,u - 2\,E(u)\right]$$

where $u = \mathrm{cn}^{-1}\left\{\frac{1-r^2}{1+r^2},\ \sin\frac{\theta}{2}\right\}$.

$$*(22)\quad \sum_{n=0}^{\infty} \frac{(-1)^n P_n}{2n+3} = \left[\frac{K}{2} - E + \cos\frac{\theta}{2}\right],$$

modulus of K and E being $k = \sin\frac{\theta}{2}$.

$$*(23)\quad \sum_{n=0}^{\infty} \frac{r^{2n-1}}{2n-1} P_n = \frac{1}{2}\left[u - 2\,\mathrm{dn}\,u\,\mathrm{cs}\,u - 2\,\mathrm{ds}\,u - 2\,E(u)\right],$$

$$u = \mathrm{cn}^{-1}\left\{\frac{1-r^2}{1+r^2},\ \cos\frac{\theta}{2}\right\}.$$

$$*(24)\quad \sum_{n=0}^{\infty} \frac{P_n}{2n-1} = \left[\frac{K}{2} - E - \sin\frac{\theta}{2}\right],$$

K and E having the same meaning as in (20).

$$*(25)\quad \sum_{n=0}^{\infty} \frac{(-1)^n r^{2n-1}}{2n-1} P_n = \frac{1}{2}\left[u - 2\,\mathrm{dn}\,u\,\mathrm{cs}\,u - 2\,\mathrm{ds}\,u - 2\,E(u)\right],$$

$$u = \mathrm{cn}^{-1}\left\{\frac{1-r^2}{1+r^2},\ \sin\frac{\theta}{2}\right\}.$$

$$*(26)\quad \sum_{n=0}^{\infty} \frac{(-1)^n P_n}{2n-1} = \left[\frac{K}{2} - E - \cos\frac{\theta}{2}\right],$$

K and E having the same meaning as in (22).

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$$(27)\quad \sum_{n=0}^{\infty} \frac{P_n}{(2n-1)(2n+3)} = -\frac{1}{2\sqrt{2}}\sqrt{1-x}$$

$$(28)\quad \sum_{n=0}^{\infty} \frac{(-1)^n P_n}{(2n-1)(2n+3)} = -\frac{1}{2\sqrt{2}}\sqrt{1+x}$$

$$(29)\quad \sum_{n=0}^{\infty} \frac{P_n}{(2n-1)(2n-3)(2n+3)(2n+5)} = \frac{1}{36\sqrt{2}}(1-x)^{3/2}$$

$$(30)\quad \sum_{n=0}^{\infty} \frac{(-1)^n P_n}{(2n-1)(2n-3)(2n+3)(2n+5)} = \frac{1}{36\sqrt{2}}(1+x)^{3/2}$$

$$*(31)\quad \sum_{n=0}^{\infty} \frac{r^{2n+5}}{2n+5} P_n = \frac{1}{3}\Bigg[r\sqrt{1-2r^2x+r^4} + 4x\left\{ \frac{u}{2} - \mathrm{dn}\,u\,\mathrm{cs}\,u + \mathrm{ds}\,u - \mathrm{E}(u) \right\} - \frac{1}{2}\left\{ \mathrm{cn}^{-1}\left(\frac{1-r^2}{1+r^2}, \cos\frac{\theta}{2} \right) \right\} \Bigg],$$

$$u = \mathrm{cn}^{-1}\left\{ \frac{1-r^2}{1+r^2}, \cos\frac{\theta}{2} \right\}$$

$$*(32)\quad \sum_{n=0}^{\infty} \frac{(-1)^n r^{2n+5}}{2n+5} P_n = \frac{1}{3}\Bigg[r\sqrt{1+2r^2x+r^4} - (4x+1)\frac{u}{2} - 4x\Big(\mathrm{ds}\,u - \mathrm{dn}\,u.\,\mathrm{cs}\,u - \mathrm{E}(u) \Big) \Bigg],\ u = \mathrm{cn}^{-1}\left\{ \frac{1-r^2}{1+r^2}, \sin\frac{\theta}{2} \right\}.$$

$$*(33)\quad \sum_{n=0}^{\infty} \frac{P_n}{2n+5} = \frac{1}{3}\left[\sqrt{2(1-x)} + 4x\left\{ \frac{\mathrm{K}}{2} - \mathrm{E} + \sin\frac{\theta}{2} \right\} - \frac{\mathrm{K}}{2} \right],$$

modulus of K and E being $\cos\frac{\theta}{2}$.

$$*(34)\quad \sum_{n=0}^{\infty} \frac{(-1)^n P_n}{2n+5} = \frac{1}{3}\left[2\cos\frac{\theta}{2} - 4\cos\theta\cos\frac{\theta}{2} - \frac{\mathrm{K}}{2}(4\cos\theta+1) + 4\cos\theta.\mathrm{E} \right],$$

modulus of K and E being $\sin\frac{\theta}{2}$.

$$*(35)\quad \sum_{n=0}^{\infty} \frac{P_n}{(2n-1)(2n+5)} = \Big[-\frac{5}{18}\sin\frac{\theta}{2} - \frac{E}{6}\Big\{1-\frac{4}{3}\cos\theta\Big\}$$

$$+\frac{K}{12}\Big\{1-\frac{1}{3}(4\cos\theta-1)\Big\} - \frac{2}{9}\cos\theta\sin\frac{\theta}{2}\Big],$$

modulus of K and E being $\cos\frac{\theta}{2}$.

$$*(36)\quad \sum_{n=0}^{\infty} \frac{(-1)^n P_n}{(2n-1)(2n+5)} = \Big[-\frac{5}{18}\cos\frac{\theta}{2} - \frac{E}{6}\Big\{1+\frac{4}{3}\cos\theta\Big\}$$

$$+\frac{K}{12}\Big\{1+\frac{1}{3}(4\cos\theta+1)\Big\} + \frac{2}{9}\cos\theta\cos\frac{\theta}{2}\Big],$$

modulus of K and E being $\sin\frac{\theta}{2}$.

$$*(37)\quad \sum_{n=0}^{\infty} \frac{P_n}{(2n+5)(2n+3)} = \frac{K}{4}\Big[1-\frac{4\cos\theta-1}{3}\Big] - \frac{E}{2}\Big[1-\frac{4}{3}\cos\theta\Big]$$

$$+\Big[\frac{1}{6}\sin\frac{\theta}{2} - \frac{2}{3}\cos\theta\sin\frac{\theta}{2}\Big],$$

modulus of K and E being $\cos\frac{\theta}{2}$.

$$*(38)\quad \sum_{n=0}^{\infty} \frac{(-1)^n P_n}{(2n+3)(2n+5)} = \frac{K}{4}\Big[1+\frac{4\cos\theta+1}{3}\Big] - \frac{E}{2}\Big[1+\frac{4}{3}\cos\theta\Big]$$

$$+\Big[\frac{1}{6}\cos\frac{\theta}{2} + \frac{2}{3}\cos\theta\cos\frac{\theta}{2}\Big],$$

modulus of K and E being $\sin\frac{\theta}{2}$.

$$(39)\quad \sum_{n=0}^{\infty} \frac{P_n}{(2n+1)(2n+5)} = \Big[\frac{K}{8}\Big\{1-\frac{4\cos\theta-1}{3}\Big\}$$

$$-\frac{1}{6}\sin\frac{\theta}{2} - \frac{1}{3}\cos\theta\sin\frac{\theta}{2} + \frac{1}{3}\cos\theta . E\Big],$$

modulus of K and E being $\cos\frac{\theta}{2}$.

$$(40)\quad \sum_{n=0}^{\infty} \frac{(-1)^n P_n}{(2n+1)(2n+5)} = \left[\frac{K}{8} \left\{ 1 + \frac{4\cos\theta+1}{3} \right\} \right.$$

$$\left. -\frac{1}{6}\cos\frac{\theta}{2} + \frac{1}{3}\cos\theta\cos\frac{\theta}{2} - \frac{1}{3}\cos\theta\cdot E \right],$$

modulus of K and E being $\sin\frac{\theta}{2}$.

$$*(41)\quad \sum_{n=0}^{\infty} \frac{r^{3n+1}}{3n+1} P_n = \left[\frac{1}{4}\wp^{-1}\left(\frac{3}{4};\, g_2, g_3\right) - \frac{1}{4}\wp^{-1}\left(\frac{3}{4};\, g'_2, g'_3\right) \right.$$

$$\left. + \frac{1}{4}\wp^{-1}\left(-\frac{H'}{X'};\, g'_2, g'_3\right) - \frac{1}{4}\wp^{-1}\left(-\frac{H}{X};\, g_2, g_3\right) \right],$$

where

$$H = -\frac{3}{4}y^4 - \frac{2x+5}{2}y^3 - \frac{18x+9}{4}y^2 - \frac{9x+3}{2}y - \frac{x^2-2x+9}{4}$$

$$H' = -\frac{3}{4}y^4 + \frac{5-2x}{2}y^3 + \frac{18x-9}{4}y^2 + \frac{3-9x}{2}y - \frac{x^2+2x+9}{4}$$

$$y = r + \frac{1}{r}$$

$$X = (y+2)(y^3-3y-2x)$$

$$X' = (y-2)(y^3-3y-2x)$$

$$g_2 = -3x + \frac{15}{4},\quad g_3 = -\frac{11}{4} + \frac{7x}{4} - \frac{x^2}{4}$$

$$g'_2 = 3\left(x + \frac{5}{4}\right),\quad g'_3 = -\frac{7x}{4} - \frac{x^2}{4} - \frac{11}{8}$$

$\wp$ being the Weierstrass's notation for the elliptic function, for example

$$\int_s^{\infty} \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}} = \wp^{-1}(s;\, g_2, g_3)$$

$$*(42)\quad \sum_{n=0}^{\infty} \frac{P_n}{3n+1} = \frac{1}{4}\wp^{-1}\left(\frac{3}{4}\ ; g_2, g_3\right) - \frac{1}{4}\wp^{-1}\left(\frac{3}{4}\ ; g'_2, g'_3\right)$$

$$-\frac{1}{4}\wp^{-1}\left(\frac{x^2+138x+185}{32(1-x)}\ ; g_2, g_3\right)$$

$$*(43)\quad \sum_{n=0}^{\infty} \frac{(-1)^n P_n}{3n+1} = \left[\frac{1}{4}\wp^{-1}\left(\frac{3}{4}\ ; g'_2 g'_3\right) - \frac{1}{4}\wp^{-1}\left(\frac{3}{4}\ ; g_2, g_3\right)\right.$$

$$\left.-\frac{1}{4}\wp^{-1}\left(\frac{x^2-138x+185}{32(1+x)}\ ; g'_2, g'_3\right)\right]$$

$$*(44)\quad \sum_{n=0}^{\infty} \frac{r^{3n+2}}{3n+2} P_n = \left[\frac{1}{4}\wp^{-1}\left(-\frac{H}{X}; g_2, g_3\right)\right.$$

$$+\frac{1}{4}\wp^{-1}\left(-\frac{H'}{X'}; g'_2, g'_3\right) - \frac{1}{4}\wp^{-1}\left(\frac{3}{4}; g_2, g_3\right)$$

$$\left.-\frac{1}{4}\wp^{-1}\left(\frac{3}{4}; g'_2, g'_3\right)\right]$$

$$*(45)\quad \sum_{n=0}^{\infty} \frac{P_n}{3n+2} = \frac{1}{4}\wp^{-1}\left(\frac{x^2+138x+185}{32(1-x)}; g_2, g_3\right)$$

$$-\frac{1}{4}\wp^{-1}\left(\frac{3}{4}\ ; g_2, g_3\right) - \frac{1}{4}\wp^{-1}\left(\frac{3}{4}\ ; g'_2, g'_3\right)$$

$$*(46)\quad \sum_{n=0}^{\infty} \frac{(-1)^n P_n}{3n+2} = \left[\frac{1}{4}\wp^{-1}\left(\frac{x^2-138x+185}{32(1+x)}\ ; g'_2, g'_3\right)\right.$$

$$\left.-\frac{1}{4}\wp^{-1}\left(\frac{3}{4}\ ; g_2, g_3\right) - \frac{1}{4}\wp^{-1}\left(\frac{3}{4}\ ; g'_2, g'_3\right)\right]$$

$$*(47)\quad \sum_{n=0}^{\infty} \frac{P_n}{(3n+1)(3n+2)} = \frac{1}{2}\wp^{-1}\left(\frac{3}{4}\ ; g_2, g_3\right)$$

$$-\frac{1}{2}\wp^{-1}\left(\frac{x^2+138x+185}{32(1-x)}\ ; g_2, g_3\right)$$

$$*(48) \quad \sum_{n=0}^{\infty} \frac{(-1^n P_n}{(3n+1)(3n+2)} = \frac{1}{2}\, \wp^{-1}\left(\frac{3}{4}\,;\, g'_2, g'_3\right)$$

$$- \frac{1}{2}\, \wp^{-1}\left(\frac{x^2-138x+185}{32(1+x)}\,;\, g'_2, g'_3\right)$$

$$*(49) \quad \sum_{n=0}^{\infty} A_n P_n = \frac{u}{4} + \frac{1}{4\cos\frac{\theta}{2}} \tan^{-1}\left(\cos\frac{\theta}{2}\,.\ \mathrm{sd}\, u\right),$$

where $\quad A_n = \dfrac{a^{2n-1}}{2n-1} - \dfrac{a^{2n-3}}{2n-3} + \dfrac{a^{2n-5}}{2n-5} -+ \dots$

$$+(-1)^n \tan^{-1}(a) + (-1)^{n-1} a \text{ and } u = \mathrm{cn}^{-1}\left(\frac{1-a^2}{1+a^2}\,,\ \cos\frac{\theta}{2}\right).$$

$$*(50) \sum_{n=0}^{\infty} \left\{ \frac{1}{2n-1} - \frac{1}{2n-3} + \frac{1}{2n-1} -+ \dots (-1)^n \left(\frac{\pi}{4} - 1\right) \right\} P_n = \frac{K}{4}$$

$$+ \frac{\pi}{8\cos\frac{\theta}{2}} - \frac{1}{8}\ \frac{\theta}{\cos\frac{\theta}{2}},$$

modulus of K being $\cos\frac{\theta}{2}$.

$$*(51) \quad \sum_{n=0}^{\infty} A_n P_n = \frac{u}{4} - \frac{1}{4\cos\frac{\theta}{2}} \tan^{-1}\left\{\cos\frac{\theta}{2}\ \ \mathrm{sd}\, u\right\},$$

where $\quad A_n = \dfrac{a^{2n+1}}{2n+1} - \dfrac{a^{2n-1}}{2n-1} + \dfrac{a^{2n-3}}{2n-3} -+ \dots (-1)^n (a - \tan^{-1} a)$

and $\quad u = \mathrm{cn}^{-1}\left(\dfrac{1-a^2}{1+a^2}\,,\ \cos\dfrac{\theta}{2}\right).$

$$*(52) \quad \sum_{n=0}^{\infty} \left[\frac{1}{2n+1} - \frac{1}{2n-1} + \frac{1}{2n-3} - \frac{1}{2n-5} + \dots \right.$$

$$\left. (-1)^n \left(1 - \frac{\pi}{4}\right) \right] P_n = \frac{K}{4} - \frac{\pi}{8\cos\frac{\theta}{2}} + \frac{\theta}{8\cos\frac{\theta}{2}},$$

modulus of K being $\cos\frac{\theta}{2}$.

$$*(53)\quad \sum_{n=0}^{\infty} A_n P_n = \frac{1}{8k}\log\left[\frac{1+k}{1-k}\cdot\frac{1-k\,\mathrm{cd}\,u}{1+k\,\mathrm{cd}\,u}\right],$$

where $\quad A_n = \dfrac{a^{2n}}{2n} - \dfrac{a^{2n-2}}{2n-2} + \dfrac{a^{2n-4}}{2n-4} -+ \ldots\ldots (-1)^{n-1}\dfrac{1}{2}a^2$

$$+\frac{(-1)^n}{2}\log(1+a^2) \text{ and } u=\mathrm{cn}^{-1}\left(\frac{1-a^2}{1+a^2},\, k\right),\ k \text{ being} = \cos\frac{\theta}{2}.$$

$$*(54)\quad \sum_{n=0}^{\infty}\left[\frac{1}{2n} - \frac{1}{2n-2} + \frac{1}{2n-4} + \ldots (-1)^{n-1}\frac{1}{2} - \frac{(-1)^n}{2}\log 2\right]P_n$$

$$= \frac{1}{8\cos\frac{\theta}{2}}\log\frac{1+\cos\frac{\theta}{2}}{1-\cos\frac{\theta}{2}}$$

$$*(55)\quad \sum_{n=0}^{\infty}\frac{r^{4n+5}}{4n+5}P_n = \frac{1}{2}\left[\left(\alpha-\frac{1}{\alpha}\right)\omega - \frac{\omega'}{\alpha} + \alpha\left\{\mathrm{dn}\,\omega'\,\mathrm{cs}\,\omega'\right.\right.$$

$$\left.\left. - \mathrm{dn}\,\omega.\,\mathrm{cs}\,\omega + E(\omega') - E(\omega)\right\}\right],$$

where $\quad \alpha = \sqrt{2+2\cos\frac{\theta}{2}},\ \beta = \sqrt{2-2\cos\frac{\theta}{2}}$

$$\omega = \mathrm{sn}^{-1}\left(\frac{\alpha}{r+\frac{1}{r}},\, \frac{\beta}{\alpha}\right) \text{ and } \omega' = K - \mathrm{sc}^{-1}\left\{\frac{\frac{1}{r}-r}{\beta},\ \sqrt{1-\frac{\beta^2}{\alpha^2}}\right\},$$

modulus of K being $\sqrt{1-\dfrac{\beta^2}{\alpha^2}}$.

$$*(56)\quad \sum_{n=0}^{\infty}\frac{P_n}{4n+5} = \frac{1}{2}\left[\left(\alpha-\frac{1}{\alpha}\right)\mathrm{sn}^{-1}\left(\frac{\alpha}{2},\frac{\beta}{\alpha}\right) + \alpha E - \frac{K}{\alpha}\right.$$

$$- \alpha E\left\{\mathrm{sn}^{-1}\left(\frac{\alpha}{2},\frac{\beta}{\alpha}\right)\right\} - \alpha\,\mathrm{dn}\left\{\mathrm{sn}^{-1}\left(\frac{\alpha}{2},\frac{\beta}{\alpha}\right)\right\}\times$$

$$\left.\mathrm{cs}\left\{\mathrm{sn}^{-1}\left(\frac{\alpha}{2},\frac{\beta}{\alpha}\right)\right\}\right],$$

modulus of K and E being $\sqrt{1-\dfrac{\beta^2}{\alpha^2}}$.

$$*(57)\quad \sum_{n=0}^{\infty} \frac{(-1)^n \mathrm{P}_n}{4n+5} = \frac{1}{2}\left[\left(\alpha' - \frac{1}{\alpha'}\right)\overset{-1}{\mathrm{sn}}\left(\frac{\alpha'}{2}, \frac{\beta'}{\alpha'}\right) + \alpha'\mathrm{E} - \frac{\mathrm{K}}{\alpha'}\right.$$

$$- \alpha'\mathrm{E}\left\{\overset{-1}{\mathrm{sn}}\left(\frac{\alpha'}{2}, \frac{\beta'}{\alpha'}\right)\right\} - \alpha'\,\mathrm{dn}\left\{\overset{-1}{\mathrm{sn}}\left(\frac{\alpha'}{2}, \frac{\beta'}{\alpha'}\right)\right\} \times$$

$$\left.\mathrm{cs}\left\{\overset{-1}{\mathrm{sn}}\left(\frac{\alpha'}{2}, \frac{\beta'}{\alpha'}\right)\right\}\right],$$

where $\alpha' = \sqrt{2+2\sin\frac{\theta}{2}}$, $\beta' = \sqrt{2-2\sin\frac{\theta}{2}}$ and modulus of K and E being $\sqrt{1-\frac{\beta'^2}{\alpha'^2}}$.

$$*(58)\quad \sum_{n=0}^{\infty} \mathrm{A}_n \mathrm{P}_n = \tfrac{1}{2}\left[\frac{1}{\alpha}\overset{-1}{\mathrm{sn}}\left(\frac{\alpha}{r+\frac{1}{r}}, \beta/\alpha\right) + \frac{\mathrm{K}}{\alpha}\right.$$

$$\left. - \frac{1}{\alpha}\overset{-1}{\mathrm{sc}}\left\{\frac{\frac{1}{r}-r}{\beta}, \sqrt{1-\frac{\beta^2}{\alpha^2}}\right\} - \frac{2\Pi(\omega', a)}{a^3 . k^2 \mathrm{sn}\, a . \mathrm{cn}\, a . \mathrm{dn}\, a}\right],$$

where $k^2 = \frac{\alpha^2-\beta^2}{\alpha^2}$, $\mathrm{sn}^2 a = \frac{\alpha^2-4}{\alpha^2-\beta^2}$, $\omega' = \mathrm{K} - \overset{-1}{\mathrm{sc}}\left\{\frac{\frac{1}{r}-r}{\beta}, k\right\}$

and $\mathrm{A}_n = \left[\frac{r^{4n-1}}{4n-1} - \frac{r^{4n-3}}{4n-3} + \ldots - r + \overset{-1}{\tan}\, r\right]$,

$\Pi(\omega', a)$ denoting the elliptic integral of the third kind.

$$*(59)\quad \sum_{n=0}^{\infty}\left[\frac{1}{4n-1} - \frac{1}{4n-3} + \ldots - 1 + \frac{\pi}{4}\right]\mathrm{P}_n = \tfrac{1}{2}\left[\frac{1}{\alpha}\overset{-1}{\mathrm{sn}}\left(\frac{\alpha}{2}, \frac{\beta}{\alpha}\right)\right.$$

$$\left. + \frac{\mathrm{K}}{\alpha} - \frac{2\Pi(\mathrm{K}, a)}{a^3 k^2 \mathrm{sn}\, a\, \mathrm{sn}\, a\, \mathrm{dn}\, a}\right]$$

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$$*(60)\quad \sum_{n=0}^{\infty} A_n P_n = \left[\frac{\Pi(\omega', a)}{a^3 k^3 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a}\right],$$

where $A_n \qquad = \left[\frac{r^{4n+1}}{4n+1} - \frac{r^{4n-1}}{4n-1} + \dots + r - \tan^{-1} r\right].$

$$*(61)\quad \sum_{n=0}^{\infty}\left[\frac{1}{4n+1} - \frac{1}{4n-1} + \cdots + 1 - \frac{\pi}{4}\right] P_n$$

$$= \left[\frac{\Pi(K, a)}{a^3 k^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a}\right].$$

$$(62)\quad \sum_{n=0}^{\infty} \frac{P_n}{(4n+3)(4n+1)} = \frac{K}{2\alpha}, \text{ modulus of K being } \sqrt{1-\frac{\beta^2}{\alpha^2}}\ .$$

$$(63)\quad \sum_{n=0}^{\infty} \frac{(-1)^n P_n}{(4n+3)(4n+1)} = \frac{K}{2\alpha'}, \text{ modulus of K being } \sqrt{1-\frac{\beta'^2}{\alpha'^2}}\ .$$

$$(64)\quad \sum_{n=0}^{\infty} \frac{P_n}{(4n+5)(4n+1)} = \frac{1}{4}\left[\frac{K}{\alpha} + \left(\frac{1}{\alpha} - \frac{\alpha}{2}\right)\operatorname{sn}^{-1}\left(\frac{\alpha}{2}, \frac{\beta}{\alpha}\right)\right.$$

$$+ \frac{\alpha}{2} E\left\{\operatorname{sn}^{-1}\left(\frac{\alpha}{2}, \frac{\beta}{\alpha}\right)\right\} - \frac{\alpha E}{2}$$

$$\left. + \frac{\alpha}{2} \operatorname{dn}\left\{\operatorname{sn}^{-1}\left(\frac{\alpha}{2}, \frac{\beta}{\alpha}\right)\right\} \operatorname{cs}\left\{\operatorname{sn}^{-1}\left(\frac{\alpha}{2}, \frac{\beta}{\alpha}\right)\right\}\right],$$

modulus of K and E being $\sqrt{1-\frac{\alpha^2}{\beta^2}}$.

$$(65)\quad \sum_{n=0}^{\infty} \frac{(-1)^n P_n}{(4n+5)(4n+1)} = \frac{1}{4}\left[\frac{K}{\alpha'} + \left(\frac{1}{\alpha'} - \frac{\alpha'}{2}\right)\operatorname{sn}^{-1}\left(\frac{\alpha'}{2}, \frac{\beta'}{\alpha'}\right)\right.$$

$$+ \frac{\alpha}{2} E\left\{\operatorname{sn}^{-1}\left(\frac{\alpha'}{2}, \frac{\beta'}{\alpha'}\right)\right\} - \frac{\alpha' E}{2}$$

$$+\frac{\alpha'}{2}\,\mathrm{dn}\left\{\overset{-1}{\mathrm{sn}}\left(\frac{\alpha'}{2},\frac{\beta'}{\alpha'}\right)\right\}\mathrm{cs}\left\{\overset{-1}{\mathrm{sn}}\left(\frac{\alpha'}{2},\frac{\beta'}{\alpha'}\right)\right\}\Bigg],$$

modulus of K and E being $\sqrt{1-\frac{\alpha'^2}{\beta'^2}}$.

$$(66)\quad \sum_{n=0}^{\infty}\frac{P_n}{(4n+5)(4n+3)}=\frac{1}{2}\left[\frac{\alpha}{2}\,E\left\{\overset{-1}{\mathrm{sn}}\left(\frac{\alpha}{2},\frac{\beta}{\alpha}\right)\right\}-\frac{\alpha E}{2}\right.$$

$$+\frac{\alpha}{2}\,\mathrm{dn}\left\{\overset{-1}{\mathrm{sn}}\left(\frac{\alpha}{2},\frac{\beta}{\alpha}\right)\right\}\mathrm{cs}\left\{\overset{-1}{\mathrm{sn}}\left(\frac{\alpha}{2},\frac{\beta}{\alpha}\right)\right\}$$

$$\left.+\left(\frac{1}{\alpha}-\frac{\alpha}{2}\right)\overset{-1}{\mathrm{sn}}\left(\frac{\alpha}{2},\frac{\beta}{\alpha}\right)\right],$$

modulus of K and E being $\sqrt{1-\frac{\beta^2}{\alpha^2}}$.

$$(67)\quad \sum_{n=0}^{\infty}\frac{(-1)^n P_n}{(4n+5)(4n+3)}=\text{Right hand side of (66) with } \alpha,\ \beta \text{ being}$$

changed into α', β'.

PART II.

Result (1).

$$\sum_{0}^{\infty}\frac{r^{3(n+1)}}{3(n+1)}\,P_n=\int_0^r\frac{t^2.\,dt}{(1-2xt^3+t^6)^{\frac{1}{2}}}$$

On integrating out, the result follows immediately.

Result (2)

can be obtained by subtraction from

$$(i)\quad \sum_{1}^{\infty}\frac{P_n}{n}=-\log\left\{\sin\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\right)\right\}$$

$$\text{and } (ii)\ \sum_{1}^{\infty}(-1)^n\frac{P_n}{n}=-\log\left\{\cos\frac{\theta}{2}\left(1+\cos\frac{\theta}{2}\right)\right\}$$

Result (3).

$$\sum_{n=0}^{\infty} \mathrm{P}_n \int_0^1 a^n(1-a)^2 da = \int_0^1 \frac{(1-a)^2 . \, da}{\sqrt{1-2ax+a^2}}$$

The result follows from integrating out both the sides.

Result (4). In (3) change x into $-x$.

Result (5).

$$\sum_0^{\infty} \mathrm{P}^n \int_0^1 a^n(1-a)^3 da = \int_0^1 \frac{(1-a)^3 da}{\sqrt{1-2ax+a^2}}$$

Result (6). Change x into $-x$ in (5).

Result (7).

$$\sum_0^{\infty} \frac{a^{n+5}}{n+5} \mathrm{P}_n = \int_0^a \frac{a^4 . \, da}{(1-2ax+a^2)^{\frac{1}{2}}}$$

The result follows by ordinary processes of Integral Calculus.

Result (8). Put $a=1$ in (7).

Result (9). Change x into $-x$ in (8).

Results (10) to (15). Put $a=i$ in results $(30)_2$ (31) and (32) of Prof. Ganesh Prasad's paper* and equate the real and imaginary parts.

Result (16). This is obtained by combining the result (2) with the well-known result (26) of Prof. Prasad's paper.

Result (17).

$$\pi \mathrm{P}_0 + \sum_1^{\infty} \mathrm{P}_n . \int_0^1 \frac{a^n}{\sqrt{a(1-a)}} = \int_0^1 \frac{da}{\sqrt{a(1-a)}\{1-2ax+a^2\}}$$

To integrate

$$\int_0^1 \frac{a^n}{\sqrt{a(1-a)}} \text{ put } a = \cos\theta$$

* *l. c*

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The integration of the right-hand side follows from a suitable transformation.†

Result (18).

Change x into $-x$ in (17)

Result (19).

$$\sum_{0}^{\infty} \frac{r^{2n+3}}{2n+3} P_n = \int_0^r \frac{t^2 \cdot dt}{(1-2xt^2+t^4)^{\frac{1}{2}}} .$$

For integration of the right-hand side we make the substitution

$$\frac{u}{2} = \int_0^t (1-2xt^2+t^4)^{-\frac{1}{2}} dt = \tfrac{1}{2} \operatorname{cn}^{-1}\left(\frac{1-t^2}{1+t^2}, \cos\frac{\theta}{2} \right)$$

and the result follows immediately ††

Result (20).

Put $r=1$ in (19)

Result (21).

Change x into $-x$ in (19)

Result (22).

Put $r=1$ in (21)

Result (23).

$$\sum_{n=1}^{\infty} \frac{r^{2n-1}}{2n-1} P_n = \int_0^r t^{-2}\{(1-2xt^2+t^4)^{-\frac{1}{2}}-1\} dt$$

To integrate the right-hand side we make the same substitution as in (19) and the result follows.

Result (24).

Put $r=1$ in (23)

Result (25).

Change x into $-x$ in 23)

Result (26).

Change x into $-x$ in (25)

‡*Result* (27).

Combine results (20) and (24)

† Greenhill's *Elliptic Functions*, p. 61.

†† See *Modern Analysis*, Whittaker and Watson, Third Edition, p. 516.

‡ These, *viz.* (27) and (29) follow also as particular cases of the expansion of $\left(\frac{1+x}{2}\right)^m$ given by Bauer, *Crelle's Journal*, Bd. 56.

Result (28).

Change x into $-x$ in (27)

Result (29).

Apply Bauer's result* to (27)

Result (30).

Change x into $-x$ in (29)

Result (31).

$$\sum_{0}^{\infty} \frac{r^{2n+5}}{2n+5} \mathrm{P}_n = \int_0^r \frac{t^4 dt}{\sqrt{1-2xt^2+t^4}}$$

we have † the reduction formula

$$x^{m-3} \sqrt{\mathrm{X}} = (m-1)au_m + 4(m-\tfrac{3}{2})bu_{m-1} + 6(m-2)cu_{m-3}$$
$$+4(m-\tfrac{5}{2})du_{m-3} + (m-3)eu_{m-4}$$

where $u_m = \int \frac{x^m \cdot dx}{\sqrt{\mathrm{X}}}$, $\mathrm{X} = ax^4 + 4bx^3 + 6cx^2 + 4dx + e$

In our case $x=t$, $a=e=1$, $b=d=0$, $6c=-2x$. The result follows immediately.

Result (32).

Change x into $-x$ in (31)

Result (33).

Put $r=1$ in (31)

Result (34).

Change x into $-x$ in (33)

Result (35).

Combine results (33) and (24)

Result (36).

Change x into $-x$ in (35)

Result (37).

Combine results (33) and (20)

Result (38).

Change x into $-x$ in (37)

* *Modern Analysis*, p. 333, Ex. 25, Third Edition.

† Greenhill's book, p. 200.

Result (39).

Combine result (33) with the Darling's result

$$K = \sum_{n=0}^{\infty} \frac{2P_n}{2n+1}.$$

Result (40).

Change x into $-x$ in (39).

Result (41).

$$\sum_{n=0}^{\infty} \frac{r^{3n+1}}{3n+1} \quad P_n = \int_0^r (1-2xt^3+t^6)^{-\frac{1}{2}} dt.$$

To integrate the right hand side we see that it is of the form

$$\int \frac{dz}{\sqrt{\alpha+\gamma z^3+\beta z^6}} \text{ where } \alpha=\beta=1,\ z=t,\ \gamma=-2x.$$

This hyperelliptic integral has been found by R. A. Roberts * to be equal to

$$\frac{1}{2k}\int \left\{ \frac{dy}{(y-2k)^{\frac{1}{2}}} - \frac{dy}{(y+2k)^{\frac{1}{2}}} \right\} \frac{1}{\sqrt{\{\beta(y^3-3k^2y+\gamma)\}}}$$

where $\alpha=\beta . k^6$ and $y=z+\frac{k^2}{z}$. In our case $k=1$ and $y=z+z^{-1}$. Thus

$\int(1-2xt^3+t^6)^{-\frac{1}{2}}dt$ has been made to depend upon the two elliptic integrals

$$(i) \quad \int \frac{dy}{\sqrt{(y-2)(y^3-3y-2x)}} \quad \text{and}$$

$$(ii) \quad \int \frac{dy}{\sqrt{(y+2)(y^3-3y-2x)}}$$

(i) and (ii) can be integrated out † easily.

Result (42).

Put $r=1$ in (41)

* *Proceedings* L. M. S., Vol. XXII, p. 32.

† See Greenhill's book, p. 62.

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Result (43).

Change x into $-x$ in (42).

Result (44).

$$\sum_{0}^{\infty} \frac{r^{3n+2}}{3n+2} \, P_n = \int_0^r (1-2xt^3+t^6)^{-\frac{1}{2}} . t . dt.$$

This hyperelliptic integral* also can be made to depend upon the two elliptic integrals (i) and (ii) of (41) and the result follows as in (41).

Result (45).

Put $r=1$ in (44).

Result (46).

Change x into $-x$ in (45)

Result (47).

Combine (41) and (42)

Result (48).

Change x into $-x$ in (47).

Result (49).

$$\sum_{n=0}^{\infty} P_n \int_0^a \frac{a^{2n} da}{1+a^2} = \int_0^a \frac{da}{(1+a^2)\sqrt{1-2a^2x+a^4}} .$$

To integrate $\int_0^a \frac{a^{2n} . da}{1+a^2}$ put $a = \tan\theta$ and we get A_n.

To integrate the right hand side we have the substitution

$$\frac{u}{2} = \int_0^a \frac{da}{\sqrt{1-2a^2x+a^4}} = \tfrac{1}{2}\,\text{cn}^{-1}\left(\frac{1-a^2}{1+a^2} \cos\frac{\theta}{2}\right) \text{ and we get}$$

$$\int_0^a \frac{da}{(1+a^2)\sqrt{1-2a^2x+a^4}} = \tfrac{1}{4}\int_0^u [1+\text{cn}\, u]du \text{ which † can be easily}$$

integrated out.

Result (50).

Put $a=1$ in (44).

* *Proceedings L. M. S.*, Vol. XXII, p. 32.

† *Modern Analysis*, p. 216. Third Edition.

Result (51).

We find that

$$\sum_0^\infty \mathrm{P}_n \int_0^a \frac{a^{2(n+1)}}{1+a^2} = \tfrac{1}{4}\int_0^u (1-\mathrm{cn}u)du = \int_0^a \frac{a^2 da}{(1+a^2)\sqrt{1-2a^2x+a^4}}$$

which integrals can be evaluated as in (49) and the result follows immediately.

Result (52).

Put $a=1$ in (51).

Result (53).

$$\sum_0^\infty \mathrm{P}_n \cdot \int_0^a \frac{a^{2n+1}}{1+a^2}\,da = \int_0^a \frac{a\,da}{(1+a^2)\sqrt{1-2a^2x+a^4}} = \tfrac{1}{4}\int_0^u \mathrm{sn}u\;du\,*$$

These integrals can be easily evaluated leading to the result.

Result (54).

Put $a=1$ in (53).

Result (55).

$$\sum_{n=0}^\infty \frac{r^{4n+5}}{4n+5}\;\mathrm{P}_n = \int_0^r \frac{r^4 dr}{\sqrt{1-2xr^4+r^8}}$$

$$\frac{r^4 dr}{\sqrt{1-2xr^4+r^8}} = \tfrac{1}{2}\left[\frac{(u^2-1)du}{\sqrt{u^4-4u^2+2(1-x)}} - \frac{(v^2+1)dv}{\sqrt{v^4+4v^2+2(1-x)}}\right]\dagger$$

where $r+\dfrac{1}{r}=u,\;\; r-\dfrac{1}{r}=v.$

Thus the hyperelliptic integrals has been made to depend upon a sum of elliptic integrals which can be easily evaluated.‡

Result (56).

Put $r=1$ in (55).

Result (57).

Change x into $-x$ in (56).

* *Modern Analysis*, p. 216, Third Edition.

† *A Tract on the Addition of Elliptic and Hyperelliptic Integrals*, by Michael Roberts, 1871, p. 59.

‡ See Greenhill's book, pp. 33 and 162, and *Modern Analysis*, p. 516.

Result (58).

$$\sum_0^\infty A_n P_n = \int_0^r \frac{dr}{(1+r^2)\sqrt{1-2xr^4+r^8}}$$

$$\frac{dr}{(1+r^2)\sqrt{1-2xr^4+r^8}} = \tfrac{1}{2}\frac{du}{\sqrt{u^4-4u^2+2(1-x)}} + \tfrac{1}{2}\frac{dv}{\sqrt{v^4+4v^2+2(1-x)}}$$

$$- \frac{dv}{(v^2+4)\sqrt{v^4+4v^2+2(1-x)}}$$

where $v=r-\dfrac{1}{r}$ and $u=r+\dfrac{1}{r}$.

The elliptic integrals can be evaluated as in (55) to give the result.

Result (59).

Put $r=1$ in (58).

Result (60).

$$\sum_0^\infty A_n P_n = \int_0^r \frac{r^2 dr}{(1+r^2)} \frac{1}{\sqrt{1-2xr^4+r^8}}$$ which can be evaluated as in (55) and (58).

Result (61).

Put $r=1$ in (60).

Result (62).

Combine results (41) and (43) of Prof. Ganesh Prasad's paper.*

Result (63).

Change x into $-x$ in (62).

Result (64).

Combine result (41) of Prof. Ganesh Prasad * with the result (56) of this paper.

Result (65).

Change x into $-x$ in (64).

Result (66).

Combine result (56) of this paper with result (42) of Prof. Ganesh Prasad.*

Result (67).

Change x into $-x$ in (66).

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* *l.c.*

Paper V

~~13~~

On the Summation of Infinite Series of Legendre's Functions.

BY

N. G. Shabde.

(*Calcutta University.*)

(*Read, the 23rd August, 1931.*)

Introduction.—The object of the present paper is to sum such infinite series of Legendre's functions, P_n (cosh σ) or Q_n (cosh σ), with non-integral n, as admit of being summed up into forms, compact and *free from the sign of integration.*

The first successful attempt at the summation of such series has been made by Professor Ganesh Prasad,* who has recently given ten such sums.

Many of the series given below are believed to be new and they are starred. There are also a number of other series, which deserve special attention, because, although they can be obtained as deductions from the results of Professor Prasad or by methods quite analogous to those used by him, their sums are so simple and interesting that they merit to be added to the list. These series are not starred.

My sincere thanks are due to Professor Ganesh Prasad for suggesting to me this continuation of his work and for his kind and keen interest in it throughout its course.

Taking $K_p(\cosh\psi) = P_{-\frac{1}{2}+pi}(\cosh\psi)$,

$$(1)\quad K_1(\cosh\psi)+\frac{1}{3}K_3(\cosh\psi)-\frac{1}{5}K_5(\cosh\psi)-\frac{1}{7}K_7(\cosh\psi)$$

$$++--++\cdots \text{ to inf.}$$

$$=\frac{1}{\sqrt{2}}\,k^{\frac{1}{2}}F(k),\ k=e^{\psi},\ \text{for } 0<\psi<\frac{\pi}{4},$$

* "On the summation of infinite series of Legendre's functions," (second paper), *Bulletin of the Calcutta Mathematical Society*, Vol. XXIII, No. 3, pp. 115-124.

and

$$= \frac{1}{\sqrt{2}}\, k^{\frac{1}{2}}\, F(k), \quad k = e^{\frac{\pi}{4}} \quad \text{for } \frac{\pi}{4} < \psi < \frac{3\pi}{4}.$$

Proof :

It is known * that

$$K_p\,(\cosh\psi) = \frac{2}{\pi}\int_0^{\psi} \frac{\cos p\phi \cdot d\phi}{\{2\,(\cosh\,\psi - \cosh\,\phi)\}^{\frac{1}{2}}}.$$

Therefore, the series in question has the sum

$$\frac{2}{\pi}\int_0^{\psi} \frac{d\phi}{\{2\,(\cosh\,\psi - \cosh\,\phi)\}^{\frac{1}{2}}}\left\{\cos\phi + \frac{1}{3}\cos 3\phi - \frac{\cos 5\phi}{5}\right.$$

$$\left. - \frac{\cos 7\phi}{7} + + \ldots\ldots \text{ to inf.}\right\}. \qquad (i)$$

The cosine † series within the crooked brackets equals $\frac{\pi}{4}\sqrt{2}$ when

$$-\frac{\pi}{4} < \phi < \frac{\pi}{4} \text{ and equals zero when } \frac{\pi}{4} < \phi < \frac{3\pi}{4}.$$

Therefore, (i) equals

$$\frac{2}{\pi}\,.\,\frac{\pi}{4}\,.\,\sqrt{2}\,.\int_0^{\psi} \frac{d\phi}{\{2\,(\cosh\psi - \cosh\phi)\}^{\frac{1}{2}}}.$$

$$= \frac{1}{\sqrt{2}}\, k^{\frac{1}{2}}\, F\,(k'), \; k = e^{\psi}, \text{ assuming } \psi \text{ to lie between zero and } \frac{\pi}{4}$$

$$\text{and } = \frac{1}{\sqrt{2}} k^{\frac{1}{2}}\, F\,(k'), \; k = e^{\frac{\pi}{4}}, \; \frac{\pi}{4} < \psi < \frac{3\pi}{4}.$$

* See Hobson's paper, "On a type of spherical harmonics of unrestricted degree, order, and argument," *Phil. Trans. of the Royal Society of London, A.*, Vol. 187, 1896, p. 530.

† *Encyklop. der Math. Wissensch.* II—I, p. 938.

(2) $$K_1(\cosh\psi)-\frac{1}{5}K_5(\cosh\psi)+\frac{1}{7}K_7(\cosh\psi)-\frac{K_{11}(\cosh\psi)}{11}+-\ldots\text{to inf.}$$

$$=\frac{1}{\sqrt{3}}k^{\frac{1}{2}}F(k'),\ k=e^{\psi},\ \text{for } 0<\psi<\frac{\pi}{3}$$

and $$=\frac{1}{\sqrt{3}}k^{\frac{1}{2}}F(k'),\ k=e^{\frac{\pi}{3}},\ \text{for } \frac{\pi}{3}<\psi<\frac{2\pi}{3}.$$

Proof:

Proceeding as in (1) we find that the series has the sum

$$\frac{2}{\pi}\int_0^{\psi}\frac{d\phi}{\{2(\cosh\psi-\cosh\phi)\}^{\frac{1}{2}}}\left\{\cos\phi-\frac{\cos 5\phi}{5}+\frac{\cos 7\phi}{7}-\frac{1}{11}\cos 11\phi+-+\ldots\text{to inf.}\right\} \qquad \text{(i)}$$

The series * within the crooked brackets has the sum $\frac{1}{\sqrt{3}}\cdot\frac{3}{2}\cdot\frac{\pi}{3}$ when $0<\phi<\frac{\pi}{3}$ and zero when $\frac{\pi}{3}<\phi<\frac{2\pi}{3}$.

Therefore (i) equals $$\frac{1}{\sqrt{3}}\int_0^{\psi}\frac{d\phi}{\{2(\cosh\psi-\cosh\phi)\}^{\frac{1}{2}}}$$

$$=\frac{1}{\sqrt{3}}k^{\frac{1}{2}}F(k'),\ k=e^{\psi}\text{ and } 0<\psi<\frac{\pi}{3}.$$

When $\frac{\pi}{3}<\psi<\frac{2\pi}{3}$, (i) equals $\frac{1}{\sqrt{3}}k^{\frac{1}{2}}F(k'),\quad k=e^{\frac{\pi}{3}}$.

(3) $$P_{-\frac{2}{3}}(\cosh\sigma)+\frac{1}{9}P_{\frac{1}{3}}(\cosh\sigma)+\frac{1\cdot 4}{9\cdot 18}P_{\frac{4}{3}}(\cosh\sigma)+\ldots\quad\text{to inf.}$$

* Whittaker and Watson, *Modern Analysis*, Third Edition, p. 192, Ex. 16.

$$=\frac{2}{6^{\frac{2}{3}}}\cdot\frac{1}{\pi}\cdot\frac{\Gamma^2(\frac{1}{6})}{\Gamma(\frac{1}{3})}\cdot\frac{1}{(3\sqrt{17}-5)^{\frac{1}{3}}}, \text{ where } \cosh\sigma=\frac{5.\sqrt{17}-3}{16}.$$

Proof:

$$^*P_{-\frac{2}{3}}(\cosh\sigma)+\frac{1}{3}\,r\,P_{\frac{1}{3}}(\cosh\sigma)+\frac{1\cdot4}{3\cdot6}\,r^2\,P_{\frac{4}{3}}(\cosh\sigma)+\ldots$$

$$\text{to inf.},\ r<e^{-\sigma}$$

$$=\frac{1}{\pi}\int_0^\pi\frac{d\phi}{(\cosh\sigma-r+\sinh\sigma\cdot\cos\phi)^{\frac{1}{3}}}$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{1}{(e^\sigma-r)^{\frac{1}{3}}}\cdot\frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{1}{3}}},\ k^2=\frac{2\sinh\sigma}{e^\sigma-r}\ (\text{Putting } \phi=2\theta)$$

$$=\frac{2}{\pi(e^\sigma-r)^{\frac{1}{3}}}\cdot\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{1}{3}}}=\frac{2}{\pi(e^\sigma-r)^{\frac{1}{3}}}\int_0^K \mathrm{dn}^{\frac{1}{3}}u\,du.$$

Put $r=\frac{1}{3}$, $k^2=\frac{3}{4}$ and hence $e^\sigma=\frac{-1+\sqrt{17}}{2}$ and $\cosh\sigma=\frac{5\sqrt{17}-3}{16}$

and we have

$$P_{-\frac{2}{3}}(\cosh\sigma)+\frac{1}{9}P_{\frac{1}{3}}(\cosh\sigma)+\frac{1\cdot4}{9\cdot18}P_{\frac{4}{3}}(\cosh\sigma)+\ldots \text{ to inf.}$$

$$=\frac{2}{\pi}\frac{6^{\frac{1}{3}}}{(3\sqrt{17}-5)^{\frac{1}{3}}}\cdot\int_0^K\sqrt[3]{\mathrm{dn}u}\,du,\ k=\frac{\sqrt{3}}{2}$$

$$=\frac{2}{\pi}\cdot\frac{6^{\frac{1}{3}}}{6\cdot(3\sqrt{17}-5)^{\frac{1}{3}}}\ \frac{\Gamma^2(\frac{1}{6})}{\Gamma(\frac{1}{3})}.$$

* Ganesh Prasad, *l. c.*, p. 123, Series IX.

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(4) $Q_{-\frac{1}{2}}(2)+\dfrac{1}{2+\sqrt{3}}\cdot\dfrac{1}{2}Q_{\frac{1}{2}}(2)+\dfrac{1\cdot3}{2\cdot4}\cdot\dfrac{1}{(2+\sqrt{3})^2}Q_{\frac{3}{2}}(2)+\ldots$ to inf.

$$=\frac{\pi}{(2\sqrt{3})^{\frac{1}{2}}}$$

Proof :

* $Q_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\,Q_{\frac{1}{2}}(\cosh\sigma)\,r+\ldots$ to inf.

$=\dfrac{2K}{\sqrt{(e^{\sigma}-r)}}$, modulus of K being $\sqrt{\left(\dfrac{e^{-\sigma}-r}{e^{\sigma}-r}\right)}$, $r<e^{\sigma}$.

Putting $\cosh\sigma=2$ and $\sqrt{\left(\dfrac{e^{-\sigma}-r}{e^{\sigma}-r}\right)}=0$,

we have $e^{\sigma}=2+\sqrt{3}$, $r=\dfrac{1}{e^{\sigma}}=\dfrac{1}{2+\sqrt{3}}$ and $K=\dfrac{\pi}{2}$.

Therefore,

$$Q_{-\frac{1}{2}}(2)+\frac{1}{2+\sqrt{3}}\,\frac{1}{2}\cdot Q_{\frac{1}{2}}(2)+\ldots \quad \text{to inf.}$$

$$=\frac{\pi}{(2\sqrt{3})^{\frac{1}{2}}}$$

(5) $Q_{-\frac{1}{2}}(\cosh 2)+\dfrac{1}{e^2}\cdot\dfrac{1}{2}\cdot Q_{\frac{1}{2}}(\cosh 2)+\dfrac{1.3}{2.4}\;\dfrac{1}{e^4}\,Q_{\frac{3}{2}}(\cosh 2)+\ldots$ to inf.

$$=\frac{\pi}{\sqrt{2\sinh 2}}$$

Proof :

Proceeding as in (4) and putting

* Ganesh Prasad, *l.c.*, p. 117, Series III.

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$\sigma=2$ and $\sqrt{\left(\dfrac{e^{-\sigma}-r}{e^{\sigma}-r}\right)}=0$, we have the required sum.

(6) $Q_{-\frac{1}{2}}(\cosh 1)+\dfrac{1}{e}\cdot\dfrac{1}{2}\cdot Q_{\frac{1}{2}}(\cosh 1)+\dfrac{1.3}{2.4}\ \dfrac{1}{e^2}\,Q_{\frac{3}{2}}(\cosh 1)$

$+\ldots$to inf.

$$=\frac{\pi}{(2\sinh 1)^{\frac{1}{2}}}$$

Proof :

Proceeding as in (4) and putting

$\sigma=1$ and $\sqrt{\left(\dfrac{e^{-\sigma}-r}{e^{\sigma}-r}\right)}=0$, we have the required sum.

(7) $(\log 2).Q_{-\frac{1}{2}}(\cosh\sigma)-Q_{\frac{1}{2}}(\cosh\sigma)+\dfrac{1}{2}\,Q_{\frac{3}{2}}(\cosh\sigma)-\dfrac{1}{3}\,Q_{\frac{5}{2}}(\cosh\sigma)$

$+\ldots$to inf.

$$=\frac{1}{\cosh\frac{\sigma}{2}}\left\{-\frac{1}{2}\,K\log\frac{1}{k}+\frac{\pi}{4}K'\right\},\ k=\operatorname{sech}\frac{\sigma}{2}.$$

Proof:

It is known* that

$$Q_{m-\frac{1}{2}}(\cosh\sigma)=\int_0^{\pi}\frac{(-1)^m\cos m\theta.d\theta}{\sqrt{2(\cosh\sigma+\cos\theta)}}.$$

Therefore the sum of the series is equal to

$$\int_0^{\pi}\frac{d\theta}{\sqrt{2}\,(\cosh\sigma+\cos\theta)}\ \left\{\log 2+\sum_1^{\infty}\frac{\cos m\theta}{m}\right\}$$

$$=-\frac{1}{2}\int_0^{\pi}\frac{\log\sin\frac{\theta}{2}\ .\ d\theta}{\sqrt{\cosh^2\frac{\sigma}{2}-\sin^2\frac{\theta}{2}}}\ ,\ \text{as}\ \log 2+\sum_1^{\infty}\frac{\cos m\theta}{m}=-\log\sin\frac{\theta}{2}$$

* Hobson, *l. c.*, p. 524.

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$$= -\int_0^{\frac{\pi}{2}} \frac{\log \sin \phi \, d\phi}{\sqrt{\cosh^2 \frac{\sigma}{2} - \sin^2 \phi}}, \quad (\text{putting } \theta = 2\phi)$$

$$= -\frac{1}{\cosh \frac{\sigma}{2}} \int_0^{\frac{\pi}{2}} \frac{\log \sin \phi . d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \quad \left(\text{where } k^2 = \text{sech}^2 \frac{\sigma}{2} \right)$$

$$= -\frac{1}{\cosh \frac{\sigma}{2}} \left\{ \frac{1}{2} \mathrm{K} \log \frac{1}{k} - \frac{\pi}{4} \mathrm{K}' \right\}.$$

(8) $(\log 2) Q_{-\frac{1}{2}} (\cosh \sigma) + Q_{\frac{1}{2}} (\cosh \sigma) + \frac{1}{2} Q_{\frac{3}{2}} (\cosh \sigma)$

$+ \frac{1}{3} Q_{\frac{5}{2}} (\cosh \sigma) + \ldots\ldots \text{to inf.}$

$$= -\frac{1}{\cosh \frac{\sigma}{2}} \left\{ \frac{\mathrm{K}}{2} \log \frac{k'}{k} - \frac{\pi}{4} \mathrm{K}' \right\}.$$

Proof:

Proceeding as in (7), the sum of the series is found to be equal to

$$-\frac{1}{\cosh \frac{\sigma}{2}} \int_0^{\frac{\pi}{2}} \frac{\log \cos \phi}{\sqrt{1 - k^2 \sin^2 \phi}} \, d\phi, \quad k^2 = \text{sech}^2 \frac{\sigma}{2}$$

$$= -\frac{1}{\cosh \frac{\sigma}{2}} \left[\frac{\mathrm{K}}{2} \log \frac{k'}{k} - \frac{1}{4} \pi \mathrm{K}' \right]$$

(9) $Q_{\frac{1}{2}} (\cosh \sigma) + \frac{1}{3} Q_{\frac{5}{2}} (\cosh \sigma) + \frac{1}{5} Q_{\frac{9}{2}} (\cosh \sigma) + \ldots$ to inf.

$$= \frac{1}{4 \cosh \frac{\sigma}{2}} \mathrm{K} \log \frac{1}{k'}, \quad k = \text{sech} \frac{\sigma}{2}$$

Proof :

Subtract (7) from (8) and the result follows immediately.

$$*(10) \quad rP_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\ \frac{r^2}{2}P_{\frac{1}{2}}(\cosh\sigma)+\frac{1.3}{2.4}\cdot\frac{r^3}{3}\ P_{\frac{3}{2}}(\cosh\sigma)$$

$$+\ldots\ldots \text{to inf.},\ r<e^{-\sigma}.$$

$$=\frac{4}{\pi}\Big[\sqrt{e^{\sigma}}\,E\Big(\sqrt{\frac{2\sinh\sigma}{e^{\sigma}}}\Big)$$

$$-\sqrt{(e^{\sigma}-r)}\,E\Big(\sqrt{\frac{2\sinh\sigma}{e^{\sigma}-r}}\Big)\Big].$$

Proof:

It is known† that

$$P_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\ P_{\frac{1}{2}}(\cosh\sigma)\ .\ r+\frac{1.3}{2.4}\ r^2P_{\frac{3}{2}}(\cosh\sigma)$$

$$+\ldots\text{to inf.},\ r<e^{-\sigma}$$

$$=\frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\sqrt{\cosh\sigma-r+\sinh\sigma\cos\phi}}.$$

Integrating this series with respect to r from $r=0$ to $r=r$ we get

$$r.P_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\ \frac{r^2}{2}\ P_{\frac{1}{2}}(\cosh\sigma)+\ \frac{1.3}{2.4}\ \frac{r^3}{3}\ P_{\frac{3}{2}}(\cosh\sigma)$$

$$+\ldots\ldots\text{to inf.}$$

$$=\frac{1}{\pi}\int_0^{\pi}d\phi\int_0^{r}\frac{dr}{\sqrt{D-r}}=-\frac{2}{\pi}\int_0^{\pi}d\phi\,\big[\sqrt{D-r}\,\big]_0^r$$

$$=\frac{2}{\pi}\int_0^{\pi}d\phi\ .\ \sqrt{D}-\frac{2}{\pi}\int_0^{\pi}d\phi.\sqrt{D-r}$$

(D standing for $\cosh\sigma+\sinh\sigma\cos\phi$)

† Ganesh Prasad, *l.c.*, p. 115.

$$=\frac{4}{\pi}\int_0^{\frac{\pi}{2}} \sqrt{e^\sigma}.\ d\theta.\ \sqrt{\left(1-\frac{2\sinh\sigma}{e^\sigma}\sin^2\theta\right)}$$

$$-\frac{4}{\pi}\int_0^{\frac{\pi}{2}} d\theta.\sqrt{e^\sigma-r}\ \sqrt{1-\left(\frac{2\sinh\sigma}{e^\sigma-r}\right)\sin^2\theta}\quad\text{(Putting }\phi=2\theta)$$

$$=\frac{4}{\pi}\left[\sqrt{e^\sigma}\mathrm{E}\left(\sqrt{\frac{2\sinh\sigma}{e^\sigma}}\right)-\sqrt{e^\sigma-r}\,\mathrm{E}\left(\sqrt{\frac{2\sinh\sigma}{e^\sigma-r}}\right)\right].$$

*(11) $\mathrm{Q}_{-\frac{1}{2}}(\cosh\sigma)+r\mathrm{Q}_{\frac{1}{2}}(\cosh\sigma)+r^2\mathrm{Q}_{\frac{3}{2}}(\cosh\sigma)+\ldots$ to inf., $|r|<1$

$$=\frac{k}{2}\left[\mathrm{K}+\frac{1-r^2}{(1+r)^2}\,\Pi\right],\ n=-\frac{4r}{(1+r)^2}\text{ and } k=\operatorname{sech}\frac{\sigma}{2}.$$

Proof :

It is known † that

$$\mathrm{Q}_{m-\frac{1}{2}}(\cosh\sigma)=\int_0^\pi\frac{(-1)^m\cos m\theta\,d\theta}{\sqrt{2(\cosh\sigma+\cos\theta)}}.$$

It is also known that

$$\frac{1+r\cos\theta}{1+2r\cos\theta+r^2}=1-r\cos\theta+r^2\cos2\theta-+\ldots\text{to inf., } |r|<1.$$

Therefore, the sum of the series in question is easily found to be equal to

$$\int_0^\pi\frac{d\theta\cdot(1+r\cos\theta)}{\{1+2r\cos\theta+r^2\}\sqrt{2(\cosh\sigma+\cos\theta)}}$$

$$=\frac{1}{2}\int^\pi\frac{d\theta}{\sqrt{2(\cosh\sigma+\cos\theta)}}+\frac{1}{2}(1-r^2)$$

$$\times\int_0^\pi\frac{1}{1+2r\cos\theta+r^2}\cdot\frac{d\theta}{\sqrt{2(\cosh\sigma+\cos\theta)}}$$

† Hobson, *l.c.*, p. 524.

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{d\phi}{(\cosh^2\frac{\sigma}{2}-\sin^2\phi)^{\frac{1}{2}}}+\frac{1}{2}\frac{(1-r^2)}{(1+r)^2}\int_0^{\frac{\pi}{2}}\frac{d\phi}{(1+n\sin^2\phi)\sqrt{\cosh^2\frac{\sigma}{2}-\sin^2\phi}}$$

(Putting $\theta=2\phi$)

$$=\frac{k}{2}\left[\mathrm{K}+\frac{(1-r^2)}{(1+r)^2}\Pi\right],\ \text{where } n=-\frac{4r}{(1+r)^2}\ \text{and } k=\operatorname{sech}\frac{\sigma}{2}.$$

*(12) $Q_{-\frac{1}{2}}(\cosh\sigma)-2rQ_{\frac{1}{2}}(\cosh\sigma)+2r^2Q_{\frac{3}{2}}(\cosh\sigma)-+\ldots$to inf. $|r|<1$

$$=\frac{1-r^2}{(1-r)^2}k\Pi,\ n=\frac{4r}{(1-r)^2}\ \text{and } k=\operatorname{sech}\frac{\sigma}{2}.$$

Proof :

$$Q_{m-\frac{1}{2}}(\cosh\sigma)=\int_0^{\pi}\frac{(-1)^m\cos m\theta\, d\theta}{\sqrt{2(\cosh\sigma+\cos\theta)}}.$$

Also

$$\frac{1-r^2}{1-2r\cos\theta+r^2}=1+2r\cos\theta+2r^2\cos2\theta+2r^3\cos3\theta+\ldots\text{to inf. } |r|<1.$$

Therefore, the required sum is equal to

$$\int_0^{\pi}\frac{(1-r^2)d\theta}{(1-2r\cos\theta+r^2)\sqrt{2(\cosh\sigma+\cos\theta)}}$$

$$=\int_0^{\frac{\pi}{2}}\frac{(1-r^2)}{(1-r)^2}\frac{d\phi}{\left(1+\frac{4r}{(1-r)^2}\sin^2\phi\right)\sqrt{\cosh^2\frac{\sigma}{2}-\sin^2\phi}}.$$

(Putting $\theta=2\phi$)

$$=\frac{1-r^2}{(1-r)^2}k\Pi$$

*(13) $P_{-\frac{1}{2}}(\cosh\sigma)+rP_{\frac{1}{2}}(\cosh\sigma)+r^2P_{\frac{3}{2}}(\cosh\sigma)+\ldots$to inf., $r<e^{-\sigma}$

$$=\frac{2}{\pi\sqrt{e^{\sigma}}}\left[\mathrm{K}+\frac{r}{e^{\sigma}-r}\Pi\right],$$

the modulus corresponding to K and Π, the complete elliptic integrals, and the parameter n corresponding to Π being respectively $\sqrt{\dfrac{2\sinh\sigma}{e^{\sigma}}}$

and $\dfrac{-2\sinh\sigma}{e^{\sigma}-r}$.

Proof :

It is known † that

$$\mathrm{P}_{m-\frac{1}{2}}(\cosh\sigma)=\frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\{\cosh\sigma+\sinh\sigma\cos\phi\}^{m+\frac{1}{2}}}.$$

Therefore the series has the sum

$$\frac{1}{\pi}\int_0^{\pi}d\phi\sum_{m=0}^{\infty}\left\{\frac{r^m}{\mathrm{D}^{m+\frac{1}{2}}}\right\},\ (\mathrm{D}\text{ standing for }\cosh\sigma+\sinh\sigma\cos\phi)$$

$$=\frac{1}{\pi}\int_0^{\pi}d\phi\,\frac{1}{\mathrm{D}^{\frac{1}{2}}}\left\{\frac{\mathrm{D}}{\mathrm{D}-r}\right\}=\frac{1}{\pi}\int_0^{\pi}d\phi\,\frac{1}{\mathrm{D}^{\frac{1}{2}}}+\frac{r}{\pi}\int_0^{\pi}\frac{d\phi}{\mathrm{D}^{\frac{1}{2}}(\mathrm{D}-r)}$$

$$=\frac{2}{\pi\sqrt{e\sigma}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}+\frac{2r}{\pi(e^{\sigma}-r)\sqrt{e\sigma}}$$

$$\times\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}},\quad\left(k^2=\frac{2\sinh\sigma}{e^{\sigma}}\text{ and }n=\frac{-2\sinh\sigma}{e^{\sigma}-r}\right)$$

(Putting $2\theta=\phi$)

$$=\frac{2}{\pi\sqrt{e\sigma}}\left[\mathrm{K}+\frac{r}{e^{\sigma}-r}\,\Pi\right]$$

*(14) $\mathrm{P}_{\frac{1}{2}}(\cosh\sigma)+2r\mathrm{P}_{\frac{3}{2}}(\text{cash }\sigma)+3r^2\mathrm{P}_{\frac{5}{2}}(\cosh\sigma)+\ldots$to inf., $r<e^{-\sigma}$

$$=\frac{2}{\pi\sqrt{e\sigma}(e^{\sigma}-r)}\left[\Pi+\frac{r.n}{(e^{\sigma}-r)\ 2(1+n)\left(1+\frac{k^2}{n}\right)}\left\{\frac{\mathrm{E}}{n}-\frac{k^2+n}{n^2}\mathrm{K}\right.\right.$$

$$\left.\left.+\left(1+\frac{2(1+k^2)}{n}+\frac{3k^2}{n^2}\right)\Pi\right\}\right],$$

† Hobson, *l.c.*, p. 524.

n and k having the same meaning as in (10).

Proof :

Differentiating (13) with respect to r, we have the required sum

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\theta}{(e^{\sigma}-r)\sqrt{e^{\sigma}}\{1+n\sin^2\theta\}\{1-k^2\sin^2\theta\}^{\frac{1}{2}}}$$

$$+\frac{2r}{\pi\sqrt{e^{\sigma}}\,(e^{\sigma}-r)^2}\int_0^{\frac{\pi}{2}} \frac{d\theta}{(1+n\sin^2\theta)^2\{1-k^2\sin^2\theta\}^{\frac{1}{2}}}$$

$$= \frac{2}{\pi\sqrt{e^{\sigma}}\,(e^{\sigma}-r)}\Bigg[\Pi+\frac{r}{(e^{\sigma}-r)}.\ -\frac{n}{2\ 1+n)(1+\frac{k^2}{n})}\left\{\frac{\mathrm{E}}{n}-\frac{k^2+n}{n^2}\mathrm{K}\right.$$

$$\left.+\left(1+\frac{2(1+k^2)}{n}+\frac{3k^2}{n^2}\right)\Pi\right\}\Bigg]$$

(For the second integral see Cayley's *Elliptic Functions*, p. 133.)

$$*(15)\quad \mathrm{P}_{\frac{1}{2}}(\cosh\sigma)+\frac{3}{2}\,r\mathrm{P}_{\frac{3}{2}}(\cosh\sigma)+\frac{3.5}{2.4}\,r^2\mathrm{P}_{\frac{5}{2}}(\cosh\sigma)+\dots$$

$$\text{to inf., } r<e^{-\sigma}$$

$$=\frac{2}{\pi}\,\frac{\mathrm{E}}{(e^{\sigma}-r)^{\frac{3}{2}}k'^2}\quad\text{where } k'^2=1-k^2=1-\frac{2\sinh\sigma}{e^{\sigma}-r}.$$

Proof:

$$\mathrm{P}_{m-\frac{1}{2}}=\frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\{\cosh\sigma+\sinh\sigma\cos\phi\}^{m+\frac{1}{2}}}.$$

Therefore the series has the sum

$$\frac{1}{\pi}\int_0^{\pi}d\phi\left\{\frac{1}{\mathrm{D}^{\frac{3}{2}}}+\frac{3}{2}\cdot\frac{r}{\mathrm{D}^{\frac{5}{2}}}+\frac{3.5}{2.4}\,\frac{r^2}{\mathrm{D}^{\frac{7}{2}}}+\dots\text{to inf.}\right\}$$

(D standing for $\cosh\sigma+\sinh\sigma\cos\phi$)

$$= \frac{1}{\pi} \int_0^{\pi} \frac{d\phi}{(D-r)^{\frac{3}{2}}}$$

$$= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\{e^{\sigma}-r-2\sinh\sigma\sin^2\theta\}^{\frac{3}{2}}} \quad \left(\text{Putting } \theta = \frac{\phi}{2}\right)$$

$$= \frac{2}{\pi(e^{\sigma}-r)^{\frac{3}{2}}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{3}{2}}}, \quad \left(k^2 = \frac{2\sinh\sigma}{e^{\sigma}-r} \right)$$

$$= \frac{2}{\pi(e^{\sigma}-r)^{\frac{3}{2}}} \int_0^{K} \frac{du}{\mathrm{dn}^2 u}$$

$$= \frac{2}{\pi(e^{\sigma}-r)^{\frac{3}{2}}} \int_0^{K} \frac{\mathrm{dn}^2 u\, du}{k'^2} \quad \text{(See \textit{Modern Analysis}, Third Edition, p. 516.)}$$

$$= \frac{2}{\pi(e^{\sigma}-r)^{\frac{3}{2}}} \frac{1}{k'^2} E, \quad k^2 = \frac{2\sinh\sigma}{e^{\sigma}-r}.$$

*(16) $Q_{\frac{1}{2}}(\cosh\sigma) + \frac{3}{2} rQ_{\frac{3}{2}}(\cosh\sigma + \frac{3.5}{2.4} r^2 Q_{\frac{5}{2}}(\cosh\sigma) + \ldots$ to inf., $r < e^{\sigma}$

$$= \frac{2}{(e^{\sigma}-r)^{\frac{3}{2}}} \left[\frac{K-E}{k^2} \right], \quad k^2 = \frac{e^{-\sigma}-r}{e^{\sigma}-r}.$$

Proof.

It is known † that

$$Q_{m-\frac{1}{2}}(\cosh\sigma) = \int_0^{\infty} \frac{d\theta}{(\cosh\sigma+\sinh\sigma\cosh\theta)^{m+\frac{1}{2}}}.$$

Therefore proceeding as in (15), we have the sum of the series equal to

$$\int_0^{\infty} \frac{d\theta}{(\cosh\sigma - r + \sinh\sigma\cosh\theta)^{\frac{3}{2}}}$$

$$= 2\int_0^{\infty} \frac{dt}{(\cosh\sigma - r+\sinh\sigma\cosh 2t)^{\frac{3}{2}}}, \text{ where } \theta = 2t$$

† See Hobson, *l.c*, p. 524.

$$=2\int_0^\infty \frac{dt}{(2\sinh\sigma\cosh^2 t - r + e^{-\sigma})^{\frac{3}{2}}}$$

$$=2\int_0^{\frac{\pi}{2}} \frac{\sec\phi\, d\phi}{(2\sinh\sigma\sec^2\phi - r + e^{-\sigma})^{\frac{3}{2}}} \text{ (where } \cosh t = \sec\phi)$$

$$=2\int_0^{\frac{\pi}{2}} \frac{\sec\phi\, d\phi}{(\sec\phi)^3\{e^{\sigma} - r - (e^{-\sigma} - r)\sin^2\phi\}^{\frac{3}{2}}}$$

$$=\frac{2}{(e^{\sigma}-r)^{\frac{3}{2}}}\int_0^{\frac{\pi}{2}} \frac{\cos^2\phi\, d\phi}{\{1-k^2\sin^2\phi\}^{\frac{3}{2}}},\ k^2 = \frac{e^{-\sigma}-r}{e^{\sigma}-r}$$

$$=\frac{2}{(e^{\sigma}-r)^{\frac{3}{2}}}\int_0^{K} \mathrm{cd}^2 u.du$$

$$=\frac{2}{(e^{\sigma}-r)^{\frac{3}{2}}}\left[\left\{\frac{u+k^2\,\mathrm{sn}\,u.\mathrm{cd}\,u}{k^2}\right\}_0^K - \frac{1}{k^2}\int_0^K \mathrm{dn}^2 u du\right]^{\dagger}$$

$$=\frac{2}{(e^{\sigma}-r)^{\frac{3}{2}}}\left[\frac{K-E}{k^2}\right],\ k^2 = \frac{e^{-\sigma}-r}{e^{\sigma}-r}.$$

*(17) $P_{\frac{3}{2}}(\cosh\sigma) + \frac{5}{2}\, r\, P_{\frac{5}{2}}(\cosh\sigma) + \frac{5\cdot 7}{2\cdot 4}\, r^2 P_{\frac{7}{2}}(\cosh\sigma)$

$+\ldots$ to inf., $r < e^{-\sigma}$

$$=\frac{2}{3\pi(e^{\sigma}-r)^{\frac{5}{2}}k'^4}\left[2(1+k'^2)E - Kk'^2\right],\ k^2 = \frac{2\sinh\sigma}{e^{\sigma}-r}.$$

Proof:

Proceeding as in (15) the sum of the series

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\{e^{\sigma} - r - 2\sinh\sigma\sin^2\theta\}^{\frac{5}{2}}}$$

$$=\frac{2}{\pi(e^{\sigma}-r)^{\frac{5}{2}}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{5}{2}}},\ k^2 = \frac{2\sinh\sigma}{e^{\sigma}-r}$$

† See Whittaker and Watson, *Modern Analysis*, Third Edition, p. 516.

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$$= \frac{2}{\pi(e^{\sigma} - r)^{\frac{5}{2}}} \int_0^{K} \frac{du}{\mathrm{dn}^4 u}$$

The above integral by means of the reduction formula †

$$(n+1)\, k'^2 \int \mathrm{nd}^{n+2}\, u\, du - n(1+k'^2) \int \mathrm{nd}^n u\, du + (n-1) \int \mathrm{nd}^{n-2}\, u\, du$$

$$+ k^2 \mathrm{nd}^{n-1}\, u.\ \mathrm{cd}u \cdot \mathrm{sd}u = 0$$

becomes

$$\frac{2}{3}\int_0^{K} \frac{1+k'^2}{k'^2} \cdot \frac{du}{\mathrm{dn}^2 u} - \frac{K}{3k'^2}\ .$$

Therefore the sum of the series is equal to

$$\frac{2}{\pi(e^{\sigma} - r)^{\frac{5}{2}}} \left[\left\{ \frac{2}{3}\ \frac{1+k'^2}{k'^2} \cdot \frac{E}{k'^2} \right\} - \frac{K}{3k'^2} \right]$$

$$= \frac{2}{3\pi(e^{\sigma} - r)^{\frac{5}{2}}\, k'^4} \left[2(1+k'^2)E - K\, k'^2 \right]$$

* (18) $\quad Q_{\frac{3}{2}}(\cosh\sigma) + \frac{5}{2}\, r\, Q_{\frac{5}{2}}(\cosh\sigma) + \frac{5\cdot 7}{2\cdot 4}\, r^2 Q_{\frac{7}{2}}(\cosh\sigma) + \ldots$

to inf., $r < e^{\sigma}$

$$= \frac{2}{3(e^{\sigma} - r)^{\frac{5}{2}} k^4} \left[2(1+k^2)(K-E) - k^2 K \right],\ k^2 = \frac{e^{-\sigma} - r}{e^{\sigma} - r}.$$

Proof :

Proceeding as in (16) we have the sum of the series equal to

$$\frac{2}{(e^{\sigma} - r)^{\frac{5}{2}}} \int_0^{K} \mathrm{cd}^4 u\, du,\quad k^2 = \frac{e^{-\sigma} - r}{e^{\sigma} - r}\ .$$

But we have the reduction formula †

† See *the Messenger of Mathematics*, Vol. XI, "On Elliptic functions" by J. W. L. Glaisher, pp. 120-133.

$$(n+1)k^2 \int \mathrm{cd}^{n+2} u\cdot du - n(1+k^2) \int \mathrm{cd}^n u\cdot du + (n-1)\int \mathrm{cd}^{n-2} u\cdot du$$
$$+ k'^2 \mathrm{cd}^{n-1} u \,\mathrm{sd}\, u \,\mathrm{nd}\, u = 0.$$

This gives

$$\int_0^K \mathrm{cd}^4 u du = \frac{2}{3}\cdot\frac{(1+k^2)}{k^2}\int_0^K \mathrm{cd}^2 u\, du - \frac{1}{3k^2}\,K$$

$$=\frac{2}{3}\;\frac{(1+k^2)}{k^2}\left[\frac{K-E}{k^2}\right] - \frac{1}{3k^2}K$$

$$=\frac{1}{3k^4}\,[2(1+k^2)\,(K-E) - K\cdot k^2].$$

Hence the sum of the series is equl to

$$\frac{2}{3(e^\sigma - r)^{\frac{5}{2}}\cdot k^4}\,[2(1+k^2)(K-E) - k^2 K]$$

$$\text{*(19)}\quad P_{\frac{5}{2}}(\cosh\sigma) + \frac{7}{2}\,r.P_{\frac{7}{2}}(\cosh\sigma) + \frac{7.9}{2.4}\,r^2 P_{\frac{9}{2}}(\cosh\sigma) +$$
$$+\ldots\text{to inf.},\ r<e^{-\sigma}$$

$$=\frac{2}{15\pi(e^\sigma - r)^{\frac{7}{2}}.k'^6}\quad[8(1+k'^2)^2 E - 4(1+k'^2)k'^2 K - 9E],\ k^2 = \frac{2\sinh\sigma}{e^\sigma - r}.$$

Proof:

Proceeding as in (17) it is easy to see that n being an odd integer

$$\text{(i)}\quad P_{\frac{n}{2}-1}(\cosh\sigma) + \frac{n}{2}\cdot r P_{\frac{n}{2}}(\cosh\sigma) + \frac{n(n+2)}{2.4}\,r^2 P_{\frac{n}{2}+1}(\cosh\sigma)$$
$$+\ldots\ldots\text{to inf.},\ r<e^{-\sigma}$$

$$=\frac{2}{\pi(e^\sigma - r)^{\frac{n}{2}}}\cdot\left[\int_0^K \mathrm{nd}^{n-1} u.du\right],\quad k^2 = \frac{2\sinh\sigma}{e^\sigma - r},\ \text{which can}$$

be evaluated by the successive use of the reduction formula†

† Glaisher, *l.c.*, pp. 120-138.

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$$k'^2(n+1)\int \mathrm{nd}^{n+2}u.du-n(1+k'^2)\int \mathrm{nd}^{n}u.du+(n-1)\int \mathrm{nd}^{n-2}u\,du$$
$$+k^2.\mathrm{nd}^{n-1}u.\mathrm{cd}u.\mathrm{sd}u=0.$$

Putting $n=7$ in (i) we have the required sum

$$=\frac{2}{\pi}\;\frac{1}{(e^{\sigma}-r)^{\frac{7}{2}}}\int_0^K \mathrm{nd}^6u\,du,\quad k^2=\frac{2\sinh\sigma}{e^{\sigma}-r}$$

$$=\frac{2}{\pi}\;\frac{1}{(e^{\sigma}-r)^{\frac{7}{2}}}\left[\frac{4(1+k'^2)}{5k'^2}\int_0^K \mathrm{nd}^4u.du-\frac{3}{5k'^2}\int_0^K \mathrm{nd}^2u.du\right]$$

$$=\frac{2}{5\pi(e^{\sigma}-r)^{\frac{7}{2}}k'^2}\left[4(1+k'^2)\left\{\frac{2}{3}\;\frac{(1+k'^2)}{k'^4}\;\mathrm{E}-\frac{\mathrm{K}}{3k'^2}\right\}-\frac{3\mathrm{E}}{k'^2}\right]$$

$$=\frac{2}{15\pi(e^{\sigma}-r)^{\frac{7}{2}}.k'^6}\;[8(1+k'^2)^2\mathrm{E}-4(1+k'^2)k'^2\mathrm{K}-9\mathrm{E}].$$

*(20) $\mathrm{Q}_{\frac{5}{2}}(\cosh\sigma)+\frac{7}{2}\,r\mathrm{Q}_{\frac{7}{2}}(\cosh\sigma)+\frac{7.9}{2.4}\,r^2.\mathrm{Q}_{\frac{9}{2}}(\cosh\sigma)$

$+\ldots\ldots$to inf., $r<e^{\sigma}$

$$=\frac{2}{15(e^{\sigma}-r)^{\frac{7}{2}}k^6}\;[8(1+k^2)^2(\mathrm{K}-\mathrm{E})-4(1+k^2)\mathrm{K}k^2-9(\mathrm{K}-\mathrm{E})],$$

$$\text{where } k^2=\frac{e^{-\sigma}-r}{e^{\sigma}-r}.$$

Proof:

Proceeding as in (18) it is easy to see that n being an odd integer

(i) $\mathrm{Q}_{\frac{n}{2}-1}(\cosh\sigma)+\frac{n}{2}\,r\mathrm{Q}_{\frac{n}{2}}(\cosh\sigma)+\frac{n(n+2)}{2.4}\,r^2\mathrm{Q}_{\frac{n}{2}+1}(\cosh\sigma)$

$+\ldots\ldots$to inf., $r<e^{\sigma}$

$=\dfrac{2}{(e^{\sigma}-r)^{\frac{n}{2}}}\left[\displaystyle\int_0^K \mathrm{cd}^{n-1}u\,du\right]$, $k^2=\dfrac{e^{-\sigma}-r}{e^{\sigma}-r}$, which can be

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evaluated by the successive use of the reduction formula

$$(n+1)k^2 \int \mathrm{cd}^{n+2}u\,du - n(1+k^2) \int \mathrm{cd}^n u\, du + (n-1) \int \mathrm{cd}^{n-2} u\,du$$
$$+ k'^2 \mathrm{cd}^{n-1}u\ .\mathrm{s}du.\ \mathrm{n}du = 0$$

Putting $n=7$ in (i) we have the required sum

$$= \frac{2}{(e^{\sigma}-r)^{\frac{7}{2}}} \int_0^K \mathrm{cd}^6 u.du$$

$$= \frac{2}{5(e^{\sigma}-r)^{\frac{7}{2}}} \left[\frac{4(1+k^2)}{k^2} \int_0^K \mathrm{cd}^4 u du - \frac{3}{k^2}. \int_0^K \mathrm{cd}^2 u.du \right]$$

$$= \frac{2}{5(e^{\sigma}-r)^{\frac{7}{2}}}. \left[\frac{4(1+k^2)}{k^2} \left\{ \frac{1}{3k^4} 2(1+k^2)(K-E) - \frac{K}{3k^2} \right\} \right.$$
$$\left. - \frac{3}{k^2} \left\{ \frac{(K-E)}{k^2} \right\} \right]$$

$$= \frac{2}{15(e^{\sigma}-r)^{\frac{7}{2}}.k^6}. [8(1+k^2)^2(K-E) - 4(1+k^2)Kk^2 - 9(K-E)].$$

*(21) $P_{-\frac{1}{4}}(\cosh\sigma) + \frac{3}{4} rP_{\frac{3}{4}}(\cosh\sigma) + \frac{3.7}{4.8} r^2 P_{\frac{7}{4}}(\cosh\sigma)$

$+ \ldots\ldots$ to inf., $r < e^{-\sigma}$

$= \dfrac{2K}{\pi(e^{\sigma}-r)^{\frac{3}{4}} \sqrt{k'}\ \sqrt{2(1+k')}}$, where the modulus corresponding

to K is $\dfrac{1-\sqrt{k'}}{\sqrt{2(1+k')}}$ and $k^2 = 1-k'^2 = \dfrac{2\sinh\sigma}{e^{\sigma}-r}$.

Proof:

Proceeding as in (15) the required sum

$$= \frac{1}{\pi} \int_0^{\pi} d\phi \left\{ \sum_{m=0}^{\infty} \frac{r^m}{D^{m+\frac{3}{4}}} \cdot \frac{3.7.11\ldots 4m-1}{4.8.12\ldots 4m} \right\}$$

$$=\frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{(D-r)^{\frac{3}{4}}}=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{(e^{\sigma}-r-2\sinh\sigma\sin^2\theta)^{\frac{3}{4}}}\quad\left(\text{putting}\ \frac{\phi}{2}=\theta\right)$$

$$=\frac{2}{\pi(e^{\sigma}-r)^{\frac{3}{4}}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{3}{4}}}\quad,\left(k^2=\frac{2\sinh\sigma}{e^{\sigma}-r}\right)$$

$$=\frac{4K}{\pi(e^{\sigma}-r)^{\frac{3}{4}}\sqrt{k'}\sqrt{2(1+k')}}$$, modulus corresponding to K being $\dfrac{1-\sqrt{k'}}{\sqrt{2(1+k')}}$. (See Greenhill's *Elliptic Functions*, p. 165.)

*(22) $\dfrac{r^2}{2}\cdot P_{-\frac{1}{2}}(\cosh\sigma)+\dfrac{1}{2}\cdot\dfrac{r^3}{3}\cdot P_{\frac{1}{2}}(\cosh\sigma)+\dfrac{1.3}{2.4}\cdot\dfrac{r^4}{4}P_{\frac{3}{2}}(\cosh\sigma)$

+......to inf., is easily summed for $r<e^{-\sigma}$.

Proof :

Multiplying each term of

$$P_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\cdot P_{\frac{1}{2}}(\cosh\sigma).r+\frac{1.3}{2.4}\ r^2P_{\frac{3}{2}}(\cosh\sigma)+\ldots\ldots\ \text{to inf.}$$

by r and integrating the series with respect to r from $r=0$ to $r=r$ we have the required sum

$$=\frac{1}{\pi}\int_0^{\pi}d\phi.\int_0^{r}\frac{rdr}{\sqrt{D-r}}$$ (D standing for $\cosh\sigma+\sinh\sigma.\cos\phi$).

Now

$$\int_0^{r}\frac{rdr}{\sqrt{D-r}}=\int_0^{r}\frac{Ddr}{\sqrt{D-r}}-\int_0^{r}\sqrt{D-r}\ dr.$$

$$=-2D\left[\sqrt{D-r}\ \right]_0^r+\frac{2}{3}\cdot\left[(D-r)^{\frac{3}{2}}\right]_0^r$$

$$=\frac{2}{3}(D-r)^{\frac{3}{2}}-2D\sqrt{D-r}+\frac{4}{3}D^{\frac{3}{2}}$$

Therefore the sum

$$=\frac{2}{\pi}\int_0^{\pi}\frac{1}{3}(\mathrm{D}-r)^{\frac{3}{2}}d\phi-\frac{2}{\pi}\int_0^{\pi}\mathrm{D}\sqrt{\mathrm{D}-r}\,d\phi+\frac{4}{3}\frac{1}{\pi}\int_0^{\pi}\mathrm{D}^{\frac{3}{2}}d\phi$$

$$=\frac{4}{3\pi}\int_0^{\frac{\pi}{2}}(1-k^2\sin^2\theta)^{\frac{3}{2}}(e^{\sigma}-r)^{\frac{3}{2}}d\theta+\frac{8}{3\pi}$$

$$\times\int_0^{\frac{\pi}{2}}(1-k_1^2\sin^2\theta)^{\frac{3}{2}}(e^{\sigma})^{\frac{3}{2}}d\theta$$

$$-\frac{4}{\pi}\int_0^{\frac{\pi}{2}}(e^{\sigma}-r)^{\frac{1}{2}}e^{\sigma}(1-k_1^2\sin^2\theta)(1-k^2\sin^2\theta)^{\frac{1}{2}}d\theta$$

(Putting $\phi=2\theta$)

where $k^2=\dfrac{2\sinh\sigma}{e^{\sigma}-r}$ and $k_1^2=\dfrac{2\sinh\sigma}{e^{\sigma}}$.

$$=\frac{4}{3\pi}(e^{\sigma}-r)^{\frac{3}{2}}\int_0^{\mathrm{F}(k)}\mathrm{dn}^4u\cdot du+\frac{8}{3\pi}(e^{\sigma})^{\frac{3}{2}}\int_0^{\mathrm{F}(k_1)}\mathrm{dn}^4v\cdot dv$$

$$-\frac{4}{\pi}(e^{\sigma}-r)^{\frac{1}{2}}e^{\sigma}.\int_0^{F(k)}(1-k_1^2\sin^2u)\,\mathrm{dn}^2u.\,du$$

where $\mathrm{dn}^2u=1-k^2\sin^2\theta$ and $\mathrm{dn}^2v=1-k_1^2\sin^2\theta$

$$=\frac{4}{9\pi}(e^{\sigma}-r)^{\frac{3}{2}}\{2(1+k'^2)\,\mathrm{E}(k)-k'^2\mathrm{F}(k)\}$$

$$+\frac{8}{9\pi}(e^{\sigma})^{\frac{3}{2}}\{2(1+k_1'^2)\mathrm{E}(k_1)-k_1'^2\mathrm{F}(k_1)\}$$

$$-\frac{4}{\pi}e^{\sigma}(e^{\sigma}-r)^{\frac{1}{2}}\int_0^{\mathrm{F}(k)}\{1+k_1^2k^2\mathrm{sn}^4u-(k^2+k_1^2)\mathrm{sn}^2u\}du$$

where $k'^2=1-k^2$ and $k_1'^2=1-k_1^2$

$$= \frac{4}{9\pi}(e^{\sigma}-r)^{\frac{3}{2}}\{2(1+k'^2)E(k)-k'^2F(k)\}$$

$$+\frac{8}{9\pi}(e^{\sigma})^{\frac{3}{2}}\{2(1+k'^2_1)E(k_1)-k'^2_1F(k_1)\}$$

$$-\frac{4}{\pi}e^{\sigma}\sqrt{(e^{\sigma}-r)}\left[F(k)-\frac{k^2+k_1^2}{k^2}\left\{F(k)-E(k)\right\}\right.$$

$$\left.+\frac{k_1^2k^2}{3k^2}\left\{\frac{2(1+k^2)}{k^2}\Big(F(k)-E(k)\Big)-F(k)\right\}\right].$$

*(23) $r\,Q_{\frac{1}{2}}(\cosh\sigma)-\frac{r^2}{2}Q_{\frac{3}{2}}(\cosh\sigma)+\frac{r^3}{3}Q_{\frac{5}{2}}(\cosh\sigma)-+\ldots$

to inf., $|r|<1$

$= \Phi$ where Φ is expressible in terms of elliptic functions.

Proof:

It is known that

$$Q_{m-\frac{1}{2}}(\cosh\sigma)=\int_0^{\pi}\frac{(-1)^m.\cos m\theta.\,d\theta}{\sqrt{2(\cosh\sigma+\cos\theta)}}.$$

Therefore, the sum of the series is equal to

$$-\int_0^{\pi}\frac{d\theta}{\sqrt{2(\cosh\sigma+\cos\theta)}}\left\{r\cos\theta+\frac{r^2}{2}\cos 2\theta+\frac{r^3}{3}\cos 3\theta+\ldots\text{to inf.}\right\}$$

$$=\int_0^{\pi}\frac{1}{2}\frac{d\theta\cdot\log(1-2r\cos\theta+r^2)}{\sqrt{2(\cosh\sigma+\cos\theta)}}$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\log\{(1-r)^2+4r\sin^2\phi\}d\phi}{\sqrt{\cosh^2\frac{\sigma}{2}-\sin^2\phi}}\qquad(\text{Putting }\theta=2\phi)$$

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$$= \frac{\log(1-r)}{\cosh\frac{\sigma}{2}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-\operatorname{sech}^2\frac{\sigma}{2}\sin^2\phi}} + \frac{1}{2\cosh\frac{\sigma}{2}}$$

$$\times\ {}^{\dagger}\!\int_0^{\frac{\pi}{2}} \frac{\log(1+n\sin^2\phi)\,d\phi}{\sqrt{1-\operatorname{sech}^2\frac{\sigma}{2}\sin^2\phi}}$$

$$= \frac{F(k)}{\cosh\frac{\sigma}{2}} \log(1-r) + \frac{1}{2\cosh\frac{\sigma}{2}} \Big[\pi F(k', a) - 2F(k)\gamma(k', a)$$

$$- \{E(k) - F(k)\}\{F(k', a)\}^2 - 2F(k)\log\sin a - \frac{\pi}{2} F(k') - F(k)\log k \Big]$$

where $k = \operatorname{sech}\frac{\sigma}{2}$, $\cot^2 a = n = \dfrac{4r}{(1-r)^2}$

and $\gamma(k, a) = {}^{\ddagger}\displaystyle\int_0^{a} \frac{E(k, a)\,da}{\sqrt{1-k^2\sin^2 a}}$

$$= \frac{E}{K}\,\frac{u^2}{2} + \log\left\{\frac{\Theta(u)}{\Theta(0)}\right\}, \text{ am } u = a$$

$$*(24) \quad P_{-\frac{5}{4}}(\cosh\sigma) - \frac{1}{4} r\, P_{-\frac{1}{4}}(\cosh\sigma) - \frac{1.3}{4.8} r^2 P_{\frac{3}{4}}(\cosh\sigma)$$

$$- \frac{1.3.7}{4.8.12} r^3 P_{\frac{7}{4}}(\cosh\sigma) - \ldots \text{to inf.}$$

is easily summed for $r < e^{-\sigma}$.

Proof:

Proceeding as in (15) the sum of the series is found to be equal to

$$\frac{2}{\pi} (e^{\sigma} - r)^{\frac{1}{4}} \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2\theta)^{\frac{1}{4}} d\theta, \quad k^2 = \frac{2\sinh\sigma}{e^{\sigma} - r}$$

† See *Liouville's Journal*, Tome XI, p. 471.

‡ See Whittaker and Watson, *Modern Analysis*, Third Edition, p. 518.

Now proceeding as for $^{*}\int_0^{\frac{\pi}{2}} \frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{1}{4}}}$ we find that

$$\int (1-k^2\sin^2\theta)^{\frac{1}{4}}d\theta = \frac{1}{2}\ \frac{1+k'}{\sqrt{1+k'}}\left[\int\frac{ydy}{\sqrt{(1-y)(1+y)(y-\beta)}}\right.$$

$$\left.+\int\frac{ydy}{\sqrt{(1-y)(1+y)(y-\beta')}}\right]+\frac{1}{2}\ \frac{\sqrt{k'}}{\sqrt{1+k'}}\left[\int\frac{dy}{\sqrt{(1-y)(1+y)(y-\beta')}}\right.$$

$$\left.-\int\frac{dy}{\sqrt{(1-y)(1+y)(y-\beta)}}\right],$$

where $y=\frac{x^4+k'}{(1+k')x^2}$, $x^2=\sqrt{1-k^2\sin^2\theta}$, $k'^2=1-k^2$,

$$\beta=\frac{2\sqrt{k'}}{1+k'}\text{ and }\beta'=\frac{-2\sqrt{k'}}{1+k'}.$$

Now making the substitution

$$\sqrt{2}u=\int_y^1\frac{dy}{\sqrt{(1-y)(1+y)(y-\beta)}}=\sqrt{2}\ \mathrm{cn}^{-1}\sqrt{\frac{y-\beta}{1-\beta}}$$

and a similar substitution

$$\sqrt{2}.\,v=\int_y^1\frac{dy}{\sqrt{(1-y)(1+y)(y-\beta')}}=\sqrt{2}\ \mathrm{cn}^{-1}\sqrt{\frac{y-\beta'}{1-\beta'}}$$

we find that

$$\int_0^{\frac{\pi}{2}}(1-k^2\sin^2\theta)^{\frac{1}{4}}d\theta$$

$$=\frac{1}{2}\cdot\frac{\sqrt{2k'}}{\sqrt{1+k'}}\Big[-2\mathrm{K}\Big]+\frac{\sqrt{2(1+k')}}{2}\left[\beta\left\{\frac{2(\mathrm{K}-\mathrm{E})}{l^2}\right\}\right.$$

$$\left.+\ \frac{2\mathrm{E}-2(1-l^2)\mathrm{K}}{l^2}\right]$$

* Greenhill, *Elliptic Functions*, p. 164.

where the modulus corresponding to K and E is $\dfrac{1-\sqrt{k'}}{\sqrt{2(1+k')}} = l$ (say).

The required sum is, therefore,

$$= \frac{2}{\pi}\ (e^{\sigma}-r)^{\frac{1}{4}}\left[\frac{\sqrt{k'}}{\sqrt{2(1+k')}}\ \left\{-2\mathrm{K}\right\} + \frac{\sqrt{2(1+k')}}{2}\ \left\{\beta\left(\frac{2(\mathrm{K}-\mathrm{E})}{l^2}\right) + \frac{2\mathrm{E}-2(1-l^2)\mathrm{K}}{l^2}\right\}\right]$$

*(25) $\mathrm{P}_{-\frac{7}{4}}(\cosh\sigma)-\dfrac{3}{4}\ r\mathrm{P}_{-\frac{3}{4}}(\cosh\sigma)-\dfrac{3.1}{4.8}\ r^2\mathrm{P}_{\frac{1}{4}}(\cosh\sigma)$

$-\dfrac{3.1.5}{4.8.12}\ r^3\mathrm{P}_{\frac{5}{4}}(\cosh\sigma)-\ldots$ to inf., $r<e^{-\sigma}$

can be easily summed up.

Proof:

Proceeding as in the last series the required sum is easily found to be

$$= \frac{2}{\pi}\ (e^{\sigma}-r)^{\frac{3}{4}}\int_0^{\frac{\pi}{2}}(1-k^2\sin^2\theta)^{\frac{3}{4}}d\theta,\ k^2=\frac{2\sinh\sigma}{e^{\sigma}-r}$$

$$= \frac{2}{\pi}\ (e^{\sigma}-r)^{\frac{3}{4}}\left[\frac{1}{2}\ \frac{\sqrt{k'}\,(1+k')}{\sqrt{1+k'}}\left\{\int_y^1\frac{ydy}{\sqrt{(1-y)(1+y)(y-\beta')}}\right.\right.$$

$$\left.-\int_y^1\frac{ydy}{\sqrt{(1-y)(1+y)(y-\beta)}}\right\}-\frac{2k'\mathrm{K}}{\sqrt{2(1+k')}}+\frac{1}{2}\ \frac{(1+k')^2}{\sqrt{1+k'}}\times$$

$$\left.\times\left\{\int_y^1\frac{y^2dy}{\sqrt{(1-y)(1+y)(y-\beta')}}+\int_y^1\frac{y^2dy}{\sqrt{(1-y)(1+y)(y-\beta)}}\right\}\right]$$

where $k'^2=1-k^2$ and the modulus corresponding to k is $l=\dfrac{1-\sqrt{k'}}{\sqrt{2(1+k')}}$

after the method used in (24).

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$$=\frac{2}{\pi}\ (e^{\sigma}-r)^{\frac{3}{4}}\Big[\frac{\sqrt{2}}{2}\ \frac{(1+k')^2}{\sqrt{1+k'}}\ \Big\{\beta^2\cdot 2\mathrm{K}+2\beta(1-\beta).\ \frac{2\mathrm{E}-2(1-l^2)\mathrm{K}}{l^2}$$

$$+\ (1-\beta)^2\left(\frac{2(2l^2-1)}{3l^4}\{2\mathrm{E}-2\mathrm{K}(1-l^2)\}+\frac{1-l^2}{3l^2}\ 2\mathrm{K}\right)\Big\}$$

$$-\ \frac{2\mathrm{K}k'}{\sqrt{2(1+k')}}-\frac{1}{2}\ \frac{\sqrt{k'}\,(1+k')}{\sqrt{1+k'}}\Big\{\beta\ \frac{2\mathrm{K}-2\mathrm{E}}{l^2}$$

$$+\ \frac{2\mathrm{E}-2(1-l^2)\mathrm{K}}{l^2}\Big\}\Big],$$

modulus corresponding to K and E being l as in (24).

* (26)

$$\frac{r^2}{2}\,\mathrm{P}_{-\frac{1}{2}}\,(\cosh\sigma)+\frac{1}{2}\cdot\frac{r^3}{2\cdot 3}\,\mathrm{P}_{\frac{1}{2}}\,(\cosh\sigma)+\frac{1\cdot 3}{2\cdot 4}\,\frac{r^4}{3\cdot 4}\,\mathrm{P}_{\frac{3}{2}}\cosh\sigma)+\ldots\text{to inf.}$$

$$r<e^{-\sigma},$$

$=\Phi$, a function expressible in terms of elliptic functions.

Proof:

Integrating n times with respect to r between zero and r the series

$$\mathrm{P}_{-\frac{1}{2}}\,(\cosh\sigma)+\frac{1}{2}\ r\cdot\mathrm{P}_{\frac{1}{2}}\,(\cosh\sigma)+\frac{1\cdot 3}{2\cdot 4}\ r^2\mathrm{P}_{\frac{3}{2}}\,(\cosh\sigma)+\ldots\text{to inf.},\ r<e^{-\sigma}$$

$$=\int_0^{\pi}\frac{d\phi}{\sqrt{\cosh\sigma-r+\sinh\sigma\cdot\cos\phi}}\text{, we see that}$$

(i) $$\frac{r^n}{1\cdot 2\cdot 3\ldots n}\cdot\mathrm{P}_{-\frac{1}{2}}\,(\cosh\sigma)+\frac{1}{2}\ \frac{r^{n+1}}{2\cdot 3\ldots(n+1)}\cdot\mathrm{P}_{\frac{1}{2}}\,(\cosh\sigma)$$

$$+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{r^{n+2}}{3.4\ldots(n+2)}\ \mathrm{P}_{\frac{3}{2}}\,(\cosh\sigma)+\ldots\ \text{to inf.},\ r<e^{-\sigma}$$

$$=\frac{1}{\pi}\left[\frac{2^2.r^{n-1}}{(n-1)!}\sqrt{e^{\sigma}}.\mathrm{E}\,(k_1)-\frac{2^3}{3}.\frac{r^{n+2}}{(n-2)!}\,(e^{\sigma})^{\frac{3}{2}}\,\frac{1}{3}\left\{2(1+k_1^2)\ \mathrm{E}(k_1)\right.\right.$$

$$\left.-k_1'^2\mathrm{F}(k_1)\right\}+\frac{2^4.\,2\,!\,.r^{n-3}}{3.5.(n-3)!}\,(e^{\sigma})^{\frac{5}{2}}\left\{\frac{4}{15}(1+k_1'^2)\Big(2(1+k_1'^2)\ \mathrm{E}(k_1)\right.$$

$$\left.-k_1'^2\ \mathrm{F}(k_1)\Big)-\frac{3}{5}\,k_1'^2\ \mathrm{E}(k_1)\right\}$$

$$+\dots(-1)^{n-1}\frac{.2^{n+1}}{3.5\dots(2n-1)}\ (e^{\sigma})^{n-\frac{1}{2}}\int_0^{\mathrm{F}(k_1)}\mathrm{dn}^{2n}v\,dv$$

$$\left.+(-1)^{n}\frac{.2^{n+1}}{3.5\dots(2n-1)}\,(e^{\sigma}-r)^{n-\frac{1}{2}}\int_0^{\mathrm{F}(k)}\mathrm{dn}^{2n}u\,du\right]$$

where $\mathrm{dn}^2v=1-k_1^2\sin^2\theta$, $\mathrm{dn}^2u=1-k^2\sin^2\theta$,

$$k_1^2=\frac{2\sinh\sigma}{e^{\sigma}}\ ,\ k^2=\frac{2\sinh\sigma}{e^{\sigma}-r}\ ,\ k_1'^2=1-k_1^2\ \text{and}\ k'^2=1-k^2,$$

the last two integrals being evaluated by the successive use of the reduction formula *

$$\int\mathrm{dn}^{2n}u\,du=\frac{2(n-1)(1+k'^2)}{2n-1}\int\mathrm{dn}^{2n-2}u.du$$

$$-\frac{(2n-3)k'^2}{2n-1}\int\mathrm{dn}^{2n-4}u\,du+\frac{1}{2n-1}\,k^2\mathrm{dn}^{2n-3}u.\mathrm{sn}\,u.\mathrm{cn}\,u=0.$$

Putting $n=2$ in (i) we have the required sum$=\Phi$

$$=\frac{4}{\pi}\,r\sqrt{e^{\sigma}}\mathrm{E}(k_1)+\frac{8}{9\pi}\,(e^{\sigma}-r)^{\frac{3}{2}}\left\{2(1+k'^2)\ \mathrm{E}(k)-k'^2\ \ \mathrm{F}\,(k)\right\}$$

$$-\frac{8}{9\pi}\,(e^{\sigma})^{\frac{3}{2}}\left\{2(1+k_1'^2)\mathrm{E}(k_1)-k_1'^2\mathrm{F}(k_1)\right\}.$$

* J. W. L. Glaisher, *l.c.*, pp. 120-138.

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A note on my first paper. †

I take this opportunity to make some ~~corrections and~~ additions in my first paper (Paper IV)

Additions.

$$* \text{ (i)} \sum_{n=0}^{\infty} \frac{(-1)^{n-1} P_{2n+1}(x)}{2(n+3)} = \left\{ \frac{2}{3} \cdot \frac{7}{4} x - \frac{5.7}{2.4} x^3 + \frac{3.3}{4.2} x \right\}$$

$$- \frac{\sqrt{x}}{4} - \frac{7}{4.3} x^{\frac{3}{2}} + \frac{\sqrt{x}}{2} \left\{ \frac{5.7}{3.4} x^2 - \frac{3}{4} \right\} + \sqrt{x} \left\{ - \frac{7.2}{4.3} x + \frac{5.7}{2.4} x^3 \right.$$

$$\left. - \frac{3.3}{2.4} x \right\} + \left\{ \frac{1}{2} \log (1+x) + \log \frac{1-\sqrt{x}}{1-x} \right\} \left\{ \frac{7x}{4} \left(\frac{5}{2} x^3 - \frac{1.5}{2.3} x - \frac{2}{3} x \right) \right.$$

$$\left. - \frac{3}{4} \left(\frac{3x^2}{2} - \frac{1}{2} \right) \right\}.$$

† "On the summation of infinite series of Legendre's polynomials," *Bulletin of the Calcutta Mathematical Society*, Vol. XXIII, No 1, pp. 23-44.

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$$* \text{(ii)} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot P_{2n}(x)}{2n+5} = -\frac{\sqrt{x}}{4} + \frac{7}{4.3} x^{\frac{3}{2}} + \frac{\sqrt{x}}{2} \left\{ \frac{5.7}{4.3} x^2 - \frac{3}{4} \right\}$$

$$-\sqrt{x} \left\{ -\frac{7.2}{4.3} x + \frac{5}{2} \cdot \frac{7}{4} \cdot x^3 - \frac{3.3}{2.4} x \right\}$$

$$+ \left\{ \frac{7x}{4} \left(\frac{5x^3}{2} - \frac{1.5}{2.3} x - \frac{2}{3} x \right) - \frac{3}{4} \left(\frac{3x^2}{2} - \frac{1}{2} \right) \right\} \tan^{-1} \frac{1}{\sqrt{x}}$$

These two series can be obtained by putting $a=i$ in the series (7)† and equating the real and imaginary parts.

† *L.c.*, p. 25.

Bull. Cal. Math. Soc., Vol. XVIII, No. 3, (1931).

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Paper VI

~~4~~

On Some Infinite Series of Legendre's Functions

BY

N. G. Shabde

Introduction.

Various infinite series for $\frac{1}{\pi}$ or $\frac{1}{\pi^2}$ have been given by Forsyth,* Hargreaves,† Glaisher,‡ and Ramanujan.§ Ramanujan also gives series for simple powers of π, for example, $\pi^{\frac{1}{2}}$ or $\frac{1}{\pi^{\frac{1}{2}}}$. We quote here three of his series by way of illustration.

$$(i)\quad 1+9\left(\frac{1}{4}\right)^4+17\left(\frac{1.5}{4.8}\right)^4+25\left(\frac{1.5.9}{4.8.12}\right)^4+\ldots=\frac{2\sqrt{2}}{\sqrt{\pi}\;(\Gamma\frac{3}{4})^2}$$

$$(ii)\quad 1-5\left(\frac{1}{2}\right)^3+9\left(\frac{1.3}{2.4}\right)^3-+\ldots=\frac{2}{\pi}$$

$$(iii)\quad 1+\frac{1}{2}\cdot\frac{1}{5^2}+\frac{1.3}{2.4}\cdot\frac{1}{9^2}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{13^2}+\ldots=\frac{(\Gamma\frac{1}{4})^2\pi}{4\sqrt{2\pi}.4}$$

* The *Messenger of Mathematics*, Vol. XII.

† The *Messenger of Mathematics*, Vol. XXVI.

‡ The *Messenger of Mathematics*, Vol. XXXVII, pp. 173-198, "On Series for $\frac{1}{\pi}$ or $\frac{1}{\pi^2}$."

§ *Collected Paper of Ramanujan* (1927). Papers 2 and 6. Also see G. H. Hardy, *Proceedings Camb. Phil. Soc.*, 21 (1923), pp. 493-503 (495). "A Chapter from Ramanujan's Note-book."

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Very recently V. Naylor and S. G. Horsley have given the series *

$$1 \operatorname{cosech} \pi - 2 \operatorname{cosech} 2\pi + 3 \operatorname{cosech} 3\pi \ldots = \frac{1}{4\pi}.$$

The object of the present paper is to give some infinite series of Legendre's functions $P_n(\cosh\sigma)$ or $Q_n(\cosh\sigma)$ with n unrestricted and σ numerical, having for their sums expressions similar to those in (*i*) and (*iii*) involving simple powers of π, for instance, π, $\frac{1}{\pi}$, $\pi^{\frac{1}{2}}$, $\frac{1}{\pi^{\frac{1}{2}}}$ or $\frac{1}{\pi^{\frac{3}{2}}}$.

(1). $P_{-\frac{2}{3}}(\cosh\sigma) + \frac{1}{3}\, r P_{\frac{1}{3}}(\cosh\sigma) + \frac{1.4}{3.6}\, r^2 P_{\frac{4}{3}}(\cosh\sigma) + \ldots$

$$= \frac{2}{\pi} \cdot \frac{1}{6} \cdot \frac{(\Gamma\frac{1}{6})^2}{\Gamma\frac{1}{3} \cdot (e^{\sigma} - r)^{\frac{1}{3}}}$$

where $r < 1$ and $e^{\sigma} = \dfrac{-3r + \sqrt{9r^2 + 16}}{2}$.

Proof :—

It is known † that

$$P_{-\frac{2}{3}}(\cosh\sigma) + \frac{1}{3}\, r P_{\frac{1}{3}}(\cosh\sigma) + \frac{1.4}{3.6}\, r^2 P_{\frac{4}{3}}(\cosh\sigma) + \ldots$$

$$= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{(e^{\sigma} - r)^{\frac{1}{3}}} \cdot \frac{d\theta}{(1 - k^2 \sin^2\theta)^{\frac{1}{3}}}, \; k^2 = \frac{2 \sinh\sigma}{e^{\sigma} - r} \text{ and } r < e^{-\sigma}.$$

* *The Journal of the London Mathematical Society*, Vol. VI, Part 3, No. 23, July 1931, " Note on the Summation of Certain Series," p. 218. In this note the sum of the series has been given as $\frac{1}{4}\pi$ instead of the correct sum $\frac{1}{4\pi}$.

† Ganesh Prasad, " On the Summation of Infinite Series of Legendre's Functions " (Second paper), *Bull. Cal. Math. Soc.*, Vol. XXIII, No. 3, pp. 115-124 (123). I take this opportunity to express my best thanks to Prof. Ganesh Prasad for suggesting to me this problem and for his kind interest and encouragement.

Putting $k^2=\frac{3}{4}$, we get $e^{\sigma}=\frac{-3r+\sqrt{9r^2+16}}{2}$

and the condition $r<e^{-\sigma}$ gives $r<1$. Again

$$\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{1}{3}}}=\int_0^{K} dn^{\frac{1}{3}}u.\, du,\ k^2=\frac{3}{4}$$

$$=\frac{1}{6}\ \frac{(\Gamma\frac{1}{6})^2}{\Gamma\frac{1}{3}}$$

Substituting these values in the series we have the required result.

$$\text{(2).}\quad Q_{\frac{1}{2}}(\cosh\sigma)+\frac{3}{2}\cdot\frac{1}{e^{\sigma}}Q_{\frac{3}{2}}(\cosh\sigma)+\frac{3.5}{2.4}\cdot\frac{1}{e^{2\sigma}}\ Q_{\frac{5}{2}}(\cosh\sigma)+\ldots$$

$$=\frac{\pi}{2}\ \frac{1}{(2\sinh\sigma)^{\frac{3}{2}}},\ \sigma>0.$$

Proof:—

It is known * that,

$$\frac{2}{(e^{\sigma}-r)^{\frac{3}{2}}}\left[\frac{K-E}{k^2}\right]=Q_{\frac{1}{2}}(\cosh\sigma)+\frac{3}{2}\ rQ_{\frac{3}{2}}(\cosh\sigma)$$

$$+\frac{3\cdot 5}{2\cdot 4}\cdot r^2Q_{\frac{5}{2}}(\cosh\sigma)+\ldots$$

$r<e^{\sigma}$ and the modulus k corresponding to the complete elliptic integrals

K and E being given by $k^2=\frac{e^{-\sigma}-r}{e^{\sigma}-r}$.

* N. G. Shabde, "On the Summation of Infinite Series of Legendre's Functions," *Bull. Cal. Math. Soc.*, Vol. XXIII, No. 3, pp. 155-182 (167).

Make $k \longrightarrow 0$. Then in the limit we have $r = \frac{1}{e^{\sigma}}$ and the condition $r < e^{\sigma}$ gives $\sigma > 0$.

Now $\lim\limits_{k \longrightarrow 0} \left[\frac{K - E}{k^2} \right] = \frac{\pi}{4}$ (*Modern Analysis* by Whittaker and Watson, 3rd edition, p. 521.)

When we substitute these values in the series, we have the required result. Numerical values can be given to σ and we shall have a number of series for $\frac{\pi}{2}$.

$$(3). \quad \frac{1}{\pi^{\frac{1}{2}}} \left\{ \frac{\Gamma\frac{1}{6}}{\Gamma\frac{2}{3} . 3^{\frac{3}{4}}} \frac{1}{(e^{\sigma} - r)^{\frac{1}{2}}} \right\} = P_{-\frac{1}{2}}(\cosh\sigma) + \frac{1}{2} r P_{\frac{1}{2}}(\cosh\sigma) + \frac{1 \cdot 3}{2 \cdot 4} r^2 P_{\frac{3}{2}}(\cosh\sigma) + \ldots, \; r < 1$$

and $$e^{\sigma} = \frac{-r(2-\sqrt{3})^2 + \sqrt{r^2(2-\sqrt{3})^4 + 16\,(2-\sqrt{3})}}{2}.$$

Proof :—

In the series *

$$\frac{2K}{\pi\sqrt{e^{\sigma} - r}} = P_{-\frac{1}{2}}(\cosh\sigma) + \frac{1}{2} r P_{\frac{1}{2}}(\cosh\sigma) + \frac{1 \cdot 3}{2 \cdot 4} r^2 P_{\frac{3}{2}}(\cosh\sigma) + \ldots,$$

$$k = \sqrt{\frac{2\sinh\sigma}{e^{\sigma} - r}} \text{ and } r < e^{-\sigma}$$

* Ganesh Prasad, l.c., p. 150.

we make $k=\sin\frac{\pi}{12}=\frac{\sqrt{3}-1}{2\sqrt{2}}$. Then we have

$$e^{\sigma}=\frac{-r(2-\sqrt{3})^2+\sqrt{r^2(2-\sqrt{3})^4+16(2-\sqrt{3})}}{2},$$

$2\mathrm{K}=\dfrac{\pi^{\frac{1}{2}}.\ \Gamma\frac{1}{6}}{3^{\frac{3}{4}}.\ \Gamma\frac{2}{3}}$ and the condition $r<e^{-\sigma}$ gives $r<1$. When we substitute these values in the series, we have the given result.

(4). $$\frac{\pi^{\frac{1}{2}}}{\sqrt{e^{\sigma}-r}}\left\{\frac{\Gamma\frac{1}{6}}{3^{\frac{3}{4}}\ \Gamma\frac{2}{3}}\right\}=\mathrm{Q}_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\,r\mathrm{Q}_{\frac{1}{2}}(\cosh\sigma)$$

$$+\frac{1\cdot 3}{2\cdot 4}\,r^2\mathrm{Q}_{\frac{3}{2}}(\cosh\sigma)+\ldots,$$

$$r<1 \text{ and } e^{\sigma}=\frac{-r(2+\sqrt{3})^2+\sqrt{r^2(2+\sqrt{3})^4+16(2+\sqrt{3})}}{2}.$$

Proof :—

In the known * series

$$\frac{2\mathrm{K}}{\sqrt{e^{\sigma}-r}}=\mathrm{Q}_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\,r\mathrm{Q}_{\frac{1}{2}}(\cosh\sigma)+\frac{1\cdot 3}{2\cdot 4}\,r^2\mathrm{Q}_{\frac{3}{2}}(\cosh\sigma)+\ldots,$$

$k=\sqrt{\dfrac{e^{-\sigma}-r}{e^{\sigma}-r}}$ and $r<e^{\sigma}$, we put $k=\sin\frac{\pi}{12}=\frac{\sqrt{3}-1}{2\sqrt{2}}$.

* Ganesh Prasad, *l.c.*, p. 117.

This gives us $2K=\dfrac{\pi^{\frac{1}{2}}\Gamma\frac{1}{6}}{3^{\frac{3}{4}}\Gamma\frac{2}{3}}$.

$$\frac{e^{-\sigma}-r}{e^{\sigma}-r}=\frac{(\sqrt{3}-1)^2}{8} \text{ gives}$$

$$e^{\sigma}=\frac{-r(2+\sqrt{3})^2+\sqrt{r^2(2+\sqrt{3})^4+16(2+\sqrt{3})}}{2}.$$

The condition $r<e^{\sigma}$ gives $r<1$. When we substitute these values in the series, we have the required result.

$$(5).\quad \frac{1}{\pi^{\frac{3}{2}}}\left\{\frac{1}{2}\left(\Gamma\frac{1}{4}\right)^2\right\}\frac{1}{(e^{\sigma}-r)^{\frac{1}{2}}}$$

$$=P_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\,rP_{\frac{1}{2}}(\cosh\sigma)+\frac{1\cdot3}{2\cdot4}\,r^2P_{\frac{3}{2}}(\cosh\sigma)+\ldots,$$

$$r<\sqrt{\frac{2}{3}} \text{ and } e^{\sigma}=\frac{-r+\sqrt{r^2+8}}{2}.$$

Proof :—

In the series

$$\frac{2}{\pi}\frac{K}{\sqrt{e^{\sigma}-r}}=P_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\,rP_{\frac{1}{2}}(\cosh\sigma)+\frac{1\cdot3}{2\cdot4}\,r^2P_{\frac{3}{2}}(\cosh\sigma)+\ldots,$$

$$k=\sqrt{\frac{2\sinh\sigma}{e^{\sigma}-r}} \text{ and } r<e^{-\sigma}$$

we put $k=\dfrac{1}{\sqrt{2}}$. This gives $e^{\sigma}=\dfrac{-r+\sqrt{r^2+8}}{2}$.

The condition $r<e^{-\sigma}$ gives $r<\sqrt{\dfrac{2}{3}}$.

Again

$$K\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{4\sqrt{\pi}}\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2.$$

When we substitute these values in the series, we have the given result.

(6). $$\frac{1}{\pi^{\frac{1}{2}}}\cdot\frac{2}{(e^{\sigma}-r)^{\frac{3}{2}}}\cdot\frac{8}{(\sqrt{3}+1)^2}\cdot\left[\frac{2.\,3^{\frac{3}{4}}.\,\Gamma\frac{2}{3}}{4.\,\sqrt{3}.\,\Gamma\frac{1}{6}}+\frac{\sqrt{3}+1}{2\sqrt{3}}\cdot\frac{\Gamma\frac{1}{6}}{2.\,3^{\frac{3}{4}}.\,\Gamma\frac{2}{3}}\right]$$

$$=P_{\frac{1}{2}}(\cosh\sigma)+\frac{3}{2}r\cdot P_{\frac{3}{2}}(\cosh\sigma)+\frac{3\cdot 5}{2\cdot 4}\cdot r^2 P_{\frac{5}{2}}(\cosh\sigma)+\ldots,$$

where $r<1$ and

$$e^{\sigma}=\frac{-r(2-\sqrt{3})^2+\sqrt{r^2(2-\sqrt{3})^4+16(2-\sqrt{3})}}{2}$$

Proof: —

It is known * that

$$\frac{2}{\pi(e^{\sigma}-r)^{\frac{3}{2}}}\cdot\frac{E}{k'^2}=P_{\frac{1}{2}}(\cosh\sigma)+\frac{3}{2}rP_{\frac{3}{2}}(\cosh\sigma)+\frac{3\cdot 5}{2\cdot 4}r^2P_{\frac{5}{2}}(\cosh\sigma)+\ldots,$$

$$r<e^{-\sigma}\text{ and }k^2=1-k'^2=\frac{2\sinh\sigma}{e^{\sigma}-r}.$$

Make $k=\sin\frac{\pi}{12}$ as in 3. We have

$$e^{\sigma}=\frac{-r(2-\sqrt{3})^2+\sqrt{r^2(2-\sqrt{3})^4+16(2-\sqrt{3})}}{2}$$

and $r<1$.

* N. G. Shabde, *l. c.*, p. 166.

Again

$$E\left(\sin\frac{\pi}{12}\right)=\pi^{\frac{1}{2}}\left[\frac{2\cdot 3^{\frac{3}{4}}.\,\Gamma_{\frac{2}{3}}}{4\sqrt{3}.\,\Gamma_{\frac{1}{6}}}+\frac{\sqrt{3}+1}{2\sqrt{3}}\cdot\frac{\Gamma_{\frac{1}{6}}}{2\cdot 3^{\frac{3}{4}}.\,\Gamma_{\frac{2}{3}}}\right].$$

By substitution we have the required result.

Conclusion.

The sums of infinite series of Legendre's functions $P_n(\cosh\sigma)$ or $Q_n(\cosh\sigma)$ with n unrestricted as given by Prof. Ganesh Prasad and the author of this paper (*Bulletin of the Calcutta Mathematical Society*, Vol. XXIII, No. 3) are seen to be function of complete elliptic integrals K, E or Π, the corresponding modulus k being itself a function of σ and another parameter r. Giving such values to k, for example, $\sin\frac{\pi}{12}$, $\tan^2\frac{\pi}{8}$ or $\frac{1}{\sqrt{2}}$, as will give for K or E expressions involving simple powers of π, many other series, similar to those given above, can be obtained.

Paper VII

* "On Some Series and Integrals Involving Associated Legendre Functions."

The object of the present paper is to give some series involving $P_n^m(\cosh\sigma)$ or $Q_n^m(\cosh\sigma)$ for unrestricted values of n and m and some integrals involving $P_n^m(z)$. Finally the value of the integral $\int_{-1}^{1} Q_n^2 \cdot d\mu$ is also noted for all values of n.

The results given below are believed to be new.

My sincere thanks are due to Professor Ganesh Prasad for the kind interest he has taken in this paper.

$$(1)\quad \frac{1}{1/2} P^1_{-\frac{1}{2}}(\cosh\sigma) + \frac{1}{2}\cdot\frac{r}{1/2}\cdot P^1_{\frac{1}{2}}(\cosh\sigma) + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{r^2}{3/2}\cdot P^1_{\frac{3}{2}}(\cosh\sigma) + \cdots \text{ to infinity.}$$

$$= \frac{2}{\pi\sqrt{e^{\sigma}-r}}\left[K - \frac{2}{k^2}(E - k'K)\right]$$

the modulus corresponding to K and E being $\sqrt{\dfrac{2\sinh\sigma}{e^{\sigma}-r}}$ and $r < e^{-\sigma}$

Proof:- It is known[1] that

$$P^m_{n-\frac{1}{2}}(\cosh\sigma) = \frac{1}{\pi}\,\frac{(-1)^m\,\Pi(n-\frac{1}{2})}{\Pi(n-m-\frac{1}{2})}\int_0^{\pi}\frac{\cos m\varphi\, d\varphi}{(\cosh\sigma + \sinh\sigma\cos\varphi)^{n+\frac{1}{2}}}$$

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(1) See Hobson's paper "On a type of Spherical Harmonies of Unrestricted degree, order and argument." (Phil. Trans. A. Vol. 187, 1896.

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* Extracted from the Bull. Calcutta Math. Soc., 25 (1933)

Therefore, the series has the sum

$$-\frac{1}{\pi}\int_0^{\pi}\frac{\cos\varphi\,d\phi}{\{\cosh\sigma - r + \sinh\sigma\cos\varphi\}^{\frac{1}{2}}}$$

$$= -\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\cos 2\theta}{\sqrt{e^{\sigma}-r}}\cdot\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\,, \quad \left(k^2 = \frac{2\sinh\sigma}{e^{\sigma}-r}\right)$$

$$= -\frac{2}{\pi}\cdot\frac{1}{\sqrt{e^{\sigma}-r}}\int_0^{\frac{\pi}{2}}\frac{2\cos^2\theta\,d\theta}{\sqrt{1-k^2\sin^2\theta}} + \frac{2}{\pi}\cdot\frac{1}{\sqrt{e^{\sigma}-r}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$= \frac{1}{\sqrt{e^{\sigma}-r}}\cdot\frac{2}{\pi}K - \frac{4}{\pi\sqrt{e^{\sigma}-r}}\int_0^{K}cn^2u\cdot du$$

$$= \frac{1}{\sqrt{e^{\sigma}-r}}\cdot\frac{2}{\pi}\cdot K - \frac{4}{\pi\sqrt{e^{\sigma}-r}}\cdot\frac{1}{k^2}\left[\int_0^{K}dn^2u\cdot du - \left(k'u^2\right)_0^{K}\right]$$

$$= \frac{2}{\pi}\cdot\frac{1}{\sqrt{e^{\sigma}-r}}K - \frac{4}{\pi\sqrt{e^{\sigma}-r}}\cdot\frac{1}{k^2}\left[E - k'^2K\right]$$

(2) $Q^{\frac{1}{2}}_{\frac{1}{2}}(\cosh\sigma) - \frac{1}{3}Q^{\frac{1}{2}}_{5/2}(\cosh\sigma) + \frac{1}{5}Q^{\frac{1}{2}}_{\frac{9}{2}}(\cosh\sigma) - + \; - - - -$ to inf.

$$= \frac{i\sqrt{2}\cdot\Pi\left(-\frac{1}{2}\right)}{4\sqrt{\sinh\sigma}}\left[\tan^{-1}\left\{\coth\frac{\sigma}{2}\right\} - \tan^{-1}\left\{\tanh\frac{\sigma}{2}\right\}\right]$$

Proof:- It is known[1] that

$$Q^{m}_{n-\frac{1}{2}}(\cosh\sigma) = (-1)^m\,\frac{2^m\,\Pi\left(m-\frac{1}{2}\right)\cdot\Pi\left(-\frac{1}{2}\right)}{}\times$$

$$\times\int_0^{\pi}\frac{\sinh^m\sigma\,.\,\cos n\phi\cdot d\phi}{(2\cosh\sigma - 2\cos\phi)^{m+\frac{1}{2}}}$$

Hence $Q^{1/2}_{n-\frac{1}{2}}(\cosh\sigma)$

$$= \frac{i\sqrt{2}\cdot\Pi\left(-\frac{1}{2}\right)}{\pi}\int_0^{\pi}\frac{\sqrt{\sinh\sigma}\cdot\cos n\phi\,d\phi}{2(\cosh\sigma - \cos\phi}$$

(1) Hobson: L.c.

Therefore the series in question has the sum

$$\frac{i\sqrt{2}\,\Pi\left(-\frac{1}{2}\right)}{\pi}\cdot\sqrt{\sinh\sigma}\cdot\int_0^{\pi}\left\{\sum_{p=1}^{\infty}\frac{(-1)^{p-1}\cdot\cos(2p-1)\,\phi}{2p-1}\right\}\frac{d\phi}{2\cosh\sigma-\cos\phi}$$

Now the cosine series with the crooked brackets equals $\frac{\pi}{4}$ or $-\frac{\pi}{4}$ according as $\cos\phi$ is $+ve$ or $-ve$; therefore, the sum

$$=\frac{i\sqrt{2}\cdot\Pi\left(-\frac{1}{2}\right)\sqrt{\sinh\sigma}}{\pi}\cdot\frac{\pi}{4}\cdot\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{2(\cosh\sigma-\cos\varphi)}-\int_0^{\frac{\pi}{2}}\frac{d\varphi}{2(\cosh\sigma+\cos\varphi)}\right]$$

$$=\frac{i\sqrt{2}\cdot\sqrt{\sinh\sigma}\cdot\Pi\left(-\frac{1}{2}\right)}{4}\left[\frac{2}{\sqrt{\cosh^2\sigma-1}}\left\{\tan^{-1}\sqrt{\frac{\cosh\sigma+1}{\cosh\sigma-1}}-\tan^{-1}\sqrt{\frac{\cosh\sigma-1}{\cosh\sigma+1}}\right\}\right]$$

$$=\frac{i\sqrt{2}\cdot\Pi\left(-\frac{1}{2}\right)}{4\sqrt{\sinh\sigma}}\left[\tan^{-1}\left\{\coth\frac{\sigma}{2}\right\}-\tan^{-1}\left\{\tanh\frac{\sigma}{2}\right\}\right]$$

(3) $K_1^{-\frac{1}{2}}(\cosh\sigma)-\frac{1}{3}\cdot K_3^{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{5}\cdot K_5^{-\frac{1}{2}}(\cosh\sigma)-+\cdots$ to inf.

$=+\frac{1}{2}\cdot\frac{\sqrt{\pi}}{\sqrt{2}}\cdot\frac{\sigma}{\sqrt{\sinh\sigma}},\quad 0<\sigma<\frac{\pi}{2}$ and taking

$$K_p^{-\frac{1}{2}}(\cosh\sigma)=P^{-\frac{1}{2}}_{-\frac{1}{2}+pi}(\cosh\sigma)$$

Proof:- It is known[1] that

$$K_p^{-m}(\cosh\sigma)=\frac{2^{1-m}\cdot\sinh^{-m}\sigma}{\Pi\left(-\frac{1}{2}\right)\Pi\left(m-\frac{1}{2}\right)}\cdot\int_0^{\sigma}\frac{\cos pu\;du}{(2\cosh\sigma-2\cosh u)^{\frac{1}{2}-m}}$$

Hence

$$K_p^{-\frac{1}{2}}(\cosh\sigma)=\frac{\sqrt{2}}{\sqrt{\sinh\sigma}\cdot\sqrt{\pi}}\int_0^{\sigma}\cos pu\cdot du$$

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(1) Hobson: l.c. or Hobson's Theory of Spherical and Ellipsoidal Harmonies. p. 270, formula 140. put $n=-\frac{1}{2}+pi$

Therefore the series has the sum

$$\sqrt{\frac{2}{\pi}}\cdot\frac{1}{\sqrt{\sinh\sigma}}\int_0^\sigma\left\{\sum_{p=1}^{\infty}\frac{(-1)^{p-1}\cos(2p-1)u}{2p-1}\right\}.\,du$$

$$=\sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{\sinh\sigma}}\cdot\frac{\pi}{4}\int_0^\sigma du, \qquad \text{taking } 0<\sigma<\frac{\pi}{2}$$

$$=\sqrt{\frac{2}{\pi}}\cdot\frac{\pi}{4\sqrt{\sinh\sigma}}\cdot\sigma = \frac{1}{2}\sqrt{\frac{\pi}{2}}\cdot\frac{\sigma}{\sqrt{\sinh\sigma}}$$

(4) Taking $T_n^m(x) = e^{\frac{1}{2}m\pi i}\cdot P_n^m(x)$

$$\sum_{n=0}^{m}(2n+1)\cdot T_n^{-\frac{1}{2}}(\cos\theta)=\frac{4}{\sqrt{2\pi\sin\theta}}\cdot\frac{\sin^2(m+1)\frac{\theta}{2}}{\sin\theta/2}$$

$$0\leq\theta<\pi$$

Proof:- It is known[1] that

$$T_n^{-\frac{1}{2}}(\cos\theta) = \frac{4\sin(n+\frac{1}{2})\theta}{(2n+1)\sqrt{2\pi\sin\theta}}$$

Hence

$$\sum_{n=0}^{m}(2n+1)\,T_n^{-\frac{1}{2}}(\cos\theta)$$

$$=\frac{4}{\sqrt{2\pi}(\sin\theta)^{\frac{1}{2}}}\left[\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}+\sin\frac{5\theta}{2}+\cdots+\sin(m+\tfrac{1}{2})\theta\right]$$

$$=\frac{4}{\sqrt{2\pi\sin\theta}}\;\frac{\sin^2\frac{(m+1)\theta}{2}}{\sin\frac{\theta}{2}}$$

(5) As it is known that $T_n^{-\frac{1}{2}}(\cos\theta)=\dfrac{4\sin(n+\frac{1}{2})\theta}{(2n+1)\sqrt{2\pi\sin\theta}}$

various sine series (Fourier) can be used to obtain infinite series of $T_n^{-\frac{1}{2}}(\cos\theta)$

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(1) MacRobert, Phil. Mag. Series 7, Vol. XIV, Oct. 1932, p. 636. Mehler - Dirichlet Integral etc."

(6) To evaluate $\int_{-1}^{1} P_n^{\mu}(z) \cdot P_m^{\mu}(z) \cdot (1-z)^{m+n} dz$.

It is known[1] that

$$\int_0^{\frac{\pi}{2}} F(-n, \mu+\beta+n; \beta; \sin^2\phi)\, F(-m, \mu+b+m; b; \sin^2\phi) \times \cos^{2\mu+1}\phi \cdot \sin^{2\mu+2\beta+2b+2m+2m-3}\phi \cdot d\phi$$

$$= (-1)^{m+n} \frac{\Gamma(\mu+m+n+1)\cdot\Gamma(\mu+\beta+b+m+n-1)(\mu+b+m)_n \times (\mu+\beta+n)_m}{2\,\Gamma(2\mu+b+\beta+2m+2n)\cdot(\beta)_n\,(b)_m}$$

Also $P_n^m(z) = \frac{1}{\Pi(-m)} \left(\frac{z+1}{z-1}\right)^{\frac{m}{2}} F\left(-n, n+1, 1-m, \frac{1-z}{2}\right)$

In the above integral we put

$\beta = b = 1-\mu$ and $\frac{1-z}{2} = \sin^2\phi$ and the integral becomes

$$\frac{1}{4}\int_{-1}^{1} [\Pi(-\mu)]^2 \left(\frac{z-1}{z+1}\right)^{\mu} P_n^{\mu}(z)\, P_m^{\mu}(z) \cdot \left(\frac{1+z}{2}\right)^{\mu} \left(\frac{1-z}{2}\right)^{m+n-\mu} dz$$

$$= (-1)^{m+n} \frac{\Gamma(\mu+m+n+1)\cdot\Gamma(1-\mu+m+n)\,(1+m)_n \times (1+n)_m}{2\,\Gamma 2(m+n+1)\quad(1-\mu)_n(1-\mu)_m}$$

or

$$\int_{-1}^{1} P_n^{\mu}(z) \cdot P_m^{\mu}(z) \cdot (1-z)^{m+n} dz$$

$$= \frac{2^{m+n+1} \cdot (-1)^{m+n-\mu}}{\{\Pi(-\mu)\}^2} \cdot \frac{\Gamma(\mu+n+1)\,\Gamma(m+n+1-\mu)\,(1+m)_n\,(1+n)_m}{\Gamma 2(m+n+1)\,(1-\mu)_n\,(1-\mu)_m}$$

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(1) W.N. Bailey: "Some definite integrals allied to an integral of Jacobi", Proc. L.M.S., Vol. 30, p. 415, result no. (2.8)

(7) To evaluate $\int_{-1}^{1} P_m^{\mu} P_n^{\mu} (1+z)^{2\mu+m+n} (1-z)^{2\mu} dz$

μ being a positive integer and m and n being unrestricted.

It is known[1] that

$$\int_{-1}^{1} (1+z)^{3\mu+m+n} (1-z)^{\mu} C_m^{\mu+\frac{1}{2}} \cdot C_n^{\mu+\frac{1}{2}} \cdot dz$$

$$=2^{4\mu+m+n+1} \frac{\Gamma(\mu+m+n+1) \cdot \Gamma(3\mu+m+n+1) \cdot \left\{\Gamma(2\mu+1)\right\}^2 \left\{\Gamma(2\mu+m+n+1)\right\}^2}{\Gamma(2\mu+m+1) \cdot \Gamma(2\mu+n+1) \cdot \Gamma(\mu+m+1) \cdot \Gamma(\mu+n+1)\ \Gamma(2(2\mu+m+n+1))}$$

Also μ being an integer

$$2\, C_{n-\mu}^{\mu+\frac{1}{2}}(z) = \frac{(z^2-1)^{-\frac{1}{2}\mu}}{(2\mu-1)(2\mu-3)\cdots 3\cdot 1} P_n^{\mu}(z)$$

So, in the above integral, putting for m and n, $m-\mu$ and $n-\mu$ respectively we have

$$\int_{-1}^{1} \frac{(1+z)^{\mu+m+n} \cdot (1-z)^{\mu} (z^2-1)^{-\mu}}{\left[(2\mu-1)(2\mu-3)\cdots 3\cdot 1\right]^2} \cdot P_m^{\mu} P_n^{\mu} \cdot dz$$

$$= \frac{2^{m+n+1} \cdot \Gamma(m+n-\mu+1)\ \Gamma(m+n+\mu+1) \left\{\Gamma(\mu+1)\right\}^2 \left\{\Gamma(m+n+1)\right\}^2}{\Gamma(m+\mu+1)\ \Gamma(\mu+n+1)\ \Gamma(m+1)\ \Gamma(n+1)\ \Gamma(2m+2n+2)}$$

Hence

$$\int_{-1}^{1} P_m^{\mu} P_n^{\mu} \cdot (1+z)^{m+n} dz$$

$$=(-1)^{\mu} \frac{\left[(2\mu-1)(2\mu-3)\cdots 3\cdot 1\right]^2 \cdot 2^{m+n+1} \left(\Gamma(\mu+1)\right)^2 \left(\Gamma(m+n+1)\right)^2 \cdot \Gamma(m+n-\mu+1)\ \Gamma(m+n+\mu+1)}{\Gamma(m+\mu+1)\ \Gamma(\mu+n+1)\ \Gamma(m+1)\ \Gamma(n+1)\ \Gamma(2m+2n+2)}$$

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(1) Bailey, l.c., result (6.2)
(2) Whittaker and Watson, <u>Modern Analysis</u>, 3rd edition p. 329.

$$(8) \quad \int_{-1}^{1} Q_n^2 \, dz = \frac{1}{2n+1}\left[\frac{\pi^2}{2} + \left\{\psi^2(n+1)\right\}(1+\cos^2 n\pi)\right]$$

$$\text{where } \psi^2(z) = \frac{d^2}{dz^2}\log\Gamma z$$

This was given by Nicholson[1] for integral values of n only.

Proof:- It is known that

$$\int_{-1}^{1} Q_m \, Q_n \, dz = \left[\frac{\left\{\psi(n+1) - \psi(m+1)\right\}(1+\cos m\pi \cdot \cos n\pi) - \frac{\pi}{2}\sin(n-m)\pi}{(n-m)(n+m+1)}\right]$$

(See Ganesh Prasad: "On non-orthogonal systems of Legendre's functions", p. 39, Vol. XII, the Proc. Benares Math. Society.)

In this integral put $m = n$. We have

$$\int_{-1}^{1} Q_n^2 \, dz = \frac{1}{2n+1}\left[\frac{\pi^2}{2}\,\underset{n=m}{Lt}\,\frac{\sin(m-n)\pi}{(m-n)\pi} + (1+\cos^2 n\pi)\,\underset{n=m}{Lt}\,\frac{\psi(n+1)-\psi(m+1)}{n-m}\right]$$

$$= \frac{1}{2n+1}\left[\frac{\pi^2}{2} + (1+\cos^2 n\pi)\cdot\psi^2(n+1)\right]$$

Note:- W.N. Bailey has recently given some interesting series and integrals involving associated Legendre functions in the Proceedings of the Cambridge Philosophical Society, Vol. XXVII parts 2 and 3. The following results may be said to be additions to those given by W. N. Bailey.

$$\text{I.} \quad \frac{(\tan\theta)^n}{(\cos\theta)^{k+1}} = \frac{2^n}{\Gamma k+n+1}\sum_{r=0}^{\infty}\frac{(n+2r)\cdot(n+r-1)!\,\Gamma n+2r+k+1}{r!}\,P_k^{-n-2r}(\cos\theta)$$

(1) Nicholson: Phil. Mag. 8 series, Vol. XLIII pp. 1 - 29. "Zonal Harmonics".

To prove this we have to replace in the series

$$\left(\frac{x}{2}\right)^n = \sum_{r=0}^{\infty} \frac{(n+2r)(n+r-1)!)}{r!} J_{n+2r}(x), \quad x \text{ by } x\sin\theta,$$

multiply $\left(\frac{1}{2}x\sin\theta\right)^k e^{-x\cos\theta}$ and integrate with regard to x from 0 to ∞, using the formula $\Gamma(\mu+\nu+1)\,P_\mu^{-\nu}(\cos\theta) = \int_0^\infty e^{-x\cos\theta} J_\nu(x\sin\theta)\,x^\mu\cdot dx$ which is valid when $R(\mu+\nu) > -1$.

II. $$\sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\left(\frac{\sin\phi}{2}\right)^m \cdot P_{m+k}^{m-\alpha}(\cos\theta) = 0$$

III $$\frac{\sin\{(k+1)\psi\}}{\{\cos^2\theta+\sin^2\theta\cos^2\phi\}^{\frac{k+1}{2}}}\left(\frac{\sin\theta}{2}\right)^{\nu}\cdot\frac{\Gamma(k+1)}{\Gamma(\nu)}$$
$$= \sum_{m=0}^{\infty}(-1)^m\cdot\Gamma(2m+k+2)\cdot(\nu+2m+1)\,C_{2m+1}^{\nu}(\cos\phi)\,P_{k-\nu}^{-\nu-2m-1}(\cos\theta)$$

IV $$\frac{\cos\{(k+1)\psi\}\left(\frac{\sin\theta}{2}\right)^{\alpha}}{\{\cos^2\theta+\sin^2\theta\cdot\cos^2\phi\}^{\frac{k+1}{2}}}\left(\frac{\sin\theta}{2}\right)^{\alpha}\cdot\frac{\Gamma(k+1)}{\Gamma(\alpha)}$$
$$= \sum_{m=0}^{\infty}(-1)^m\cdot(\alpha+2m)\cdot\Gamma(k+2m+1)\cdot C_{2m}^{\alpha}(\cos\phi)\,P_{k-\alpha}^{-\alpha-2m}(\cos\theta)$$

where $\tan\psi = \tan\theta\cdot\cos\phi$

Chapter II
Some Properties of associated Legendre Functions.
Paper I

ON VARIOUS RECURRENCE FORMULAE FOR $P_n^m(z)$ AND $Q_n^m(z)$ WITH UNRESTRICTED VALUES OF n, m AND z.

By

N. G. SHABDE.

Introduction.—The object of this paper is to generalize *for unrestricted n, m and z* the various recurrence formulae for $P_n^m(z)$ and $Q_n^m(z)$, which were given by Heine* for integral m and n and by F. Neumann† for restricted n, m and z.

Prof. Hobson‡ has proved four recurrence formulae for $P_n^m(z)$ or $Q_n^m(z)$ with unrestricted n, m and z. Of these four formulae, the fundamental difference equation, viz,

$$(2n+1)\,z\,P_n^m(z)-(n-m+1)\,P_{n+1}^m(z)-(n+m)\,P_{n-1}^m(z)=o,$$

has been proved by Prof. G. Prasad by a simple method in his recent lectures on "Spherical Harmonics" delivered at the Calcutta University. This method has been applied in this paper to prove the remaining three recurrence formulae given by Prof. Hobson and also to generalize the formulae given by Heine and Neumann.

§ 1 contains the formulae proved by Prof. Hobson. In § 2 are generalized the formulae given by Heine and in § 3 those given by Neumann. Throughout the paper n, m and z should be taken to be *unrestricted.*

I take this opportunity to thank Prof. Ganesh Prasad for kindly suggesting this work and for his interest and advice throughout its course.

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* *Handbuch der Kugelfunktionen*, Bd I, 1878, pp. 258-259.

† *Beiträge zur Theorie der Kugelfunktionen* von Dr. F. Neumann, 1878, pp. 74-77.

‡ "On a type of Spherical Harmonics of unrestricted degree, order and argument" by E. W. Hobson, *Philosophical Transactions of the Royal Society of London*, Vol. 187 (1896), A, pp. 443-531; see specially pp. 521-523.

Prof. Hobson* defines $P_n^m(z)$ and $Q_n^m(z$ for unrestricted n, m and z by the following expressions :—

$$P_n^m(z) = \frac{e^{-n\pi i}}{4\pi \sin n\pi} \cdot \frac{1}{2^n} \cdot \frac{\Pi(n+m)}{\Pi(n)} \times$$

$$(z^2-1)^{\frac{m}{2}} \int^{(z+,\,1+,\,z-,\,1-)} (t^2-1)^n (t-z)^{-n-m-1}\, dt$$

and

$$Q_n^m(z) = \frac{e^{-(n+1)\pi i}}{4i \sin n\pi} \cdot \frac{\Pi(n+m)}{\Pi(n)} \times$$

$$(z^2-1)^{\frac{m}{2}} \int^{(-1+,\,1-)} \frac{1}{2^n} (t^2-1)^n (t-z)^{-n-m-1}\, dt$$

the contours in the two cases being respectively as those shown in figures I and II below, a cross-cut being supposed to be made along the real axis from 1 to $-\infty$, the phases of the expressions in the integrands being definitely assigned at the starting point of the contour and $\Pi(x)$ being the Gaussian function and equal to $\Gamma(x+1)$.

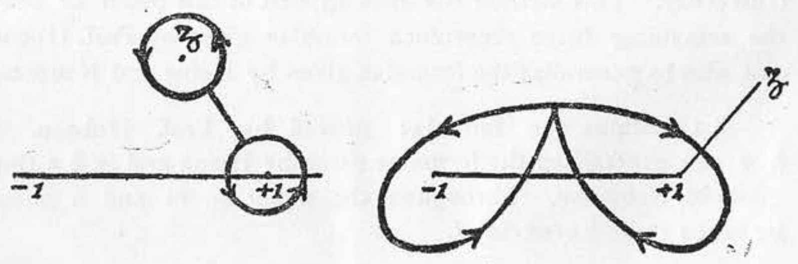

Fig. I Fig. II

We shall prove the recurrence formulae for $P_n^m(z)$ and, as it is easily seen, the proofs apply word for word for $Q_n^m(z)$ except with the necessary difference in the contour taken and the necessary change in the constant term multiplying the integrand.

* *l. c.* pp. 451 and 455.

§ 1

I. (*a*). *To prove*

$$P_n^{m+2}(z)+2(m+1)\frac{z}{\sqrt{(z^2-1)}}P_n^{m+1}(z)$$
$$-(n-m)(n+m+1)P_n^m(z)$$
$$=0.$$

Proof:—Let the left-hand side of (*a*) be denoted by $\phi(z)$.

Then

$$\phi(z)=\frac{e^{-n\pi i}}{4\pi\sin n\pi}\frac{1}{2^n}\cdot\frac{\Pi(n+m+1)}{\Pi(n)}(z^2-1)^{\frac{m}{2}}\int^{(z+,1+,z-,1-,)} I\,dt$$

where

$$I=\Big[(n+m+2)\frac{(t^2-1)^n.(z^2-1)}{(t-z)^{n+m+3}}$$
$$+\frac{2(m+1)z(t^2-1)^n}{(t-z)^{n+m+2}}-(n-m)\frac{(t^2-1)^n}{(t-z)^{n+m+1}}\Big]$$
$$=\Big[\frac{(n+m+2)(t^2-1)^{n+1}}{(t-z)^{n+m+3}}-\frac{(n+m+2)(t^2-1)^n}{(t-z)^{n+m+1}}$$
$$-(n-m)\frac{(t^2-1)^n}{(t-z)^{n+m+1}}+\Big\{2(m+1)-2(n+m+2)\Big\}\frac{z(t^2-1)^n}{(t-z)^{n+m+2}}\Big],$$

since we can write (z^2-1) as $\Big\{(t^2-1)-(t-z)(t-z+2z)\Big\}$.

So

$$I=\Big[\frac{(n+m+2)(t^2-1)^{n+1}}{(t-z)^{n+m+3}}-2(n+1)\frac{(t^2-1)^n}{(t-z)^{n+m+1}}$$
$$-\frac{2z(n+1)(t^2-1)^n}{(t-z)^{n+m+2}}\Big]=-\frac{d}{dt}\Big\{\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+2}}\Big\}$$

$\therefore\ \phi(z)=$

$$-\frac{e^{-n\pi i}}{4\pi\sin n\pi}\frac{1}{2^n}\cdot\frac{\Pi(n+m+1)}{\Pi(n)}\times$$

$$(z^2-1)^{\frac{m}{2}}\int^{(z+,1+,z-,1-)}\frac{d}{dt}\Big\{\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+2}}\Big\}dt=0.$$

1 (*b*) *To prove*

$$Q_n^{m+2}(z)+2(m+1)\frac{z}{\sqrt{(z^2-1)}}Q_n^{m+1}(z)-(n-m)(n+m+1)Q_n^m(z)$$

$$=0$$

Proof:—Exactly as in (*a*), if $\phi(z)$ denotes the left-hand side of (*b*), then $\phi(z)$

$$=-\frac{e^{-(n+1)\pi i}}{4i\sin n\pi}\cdot\frac{\Pi(n+m+1)}{\Pi(n)}\cdot(z^2-1)^{\frac{m}{2}}\cdot\frac{1}{2^n}\int^{(-1+,\,1-)}\frac{d}{dt}\left\{\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+2}}\right\}dt$$

$$=0$$

2 (*a*) *To prove*

$$(z^2-1)\frac{d\,P_n^m(z)}{dz}=nz\,P_n^m(z)-(n+m)\,P_{n-1}^m(z)$$

Proof:—Let $\phi(z)$ denote $(z^2-1)\dfrac{d\,P_n^m(z)}{dz}-nz\,P_n^m(z)$

$$+(n+m)\,P_{n-1}^m(z).$$

Then $\phi(z)$

$$=\frac{e^{-n\pi i}}{4\pi\sin n\pi}\;\frac{\Pi(n+m)}{\Pi(n)}\cdot\frac{1}{2^n}(z^2-1)^{\frac{m}{2}}\int^{(z+,\,1+,\,z-,\,1-)}\mathrm{I}\,dt$$

where

$$\mathrm{I}=\Big[(n+m+1)\left\{\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+2}}-\frac{(t^2-1)^n}{(t-z)^{n+m}}-\frac{2z(t^2-1)^n}{(t-z)^{n+m+1}}\right\}$$

$$+\frac{mz(t^2-1)^n}{(t-z)^{n+m+1}}-\frac{nz(t^2-1)^n}{(t-z)^{n+m+1}}+\frac{2n(t^2-1)^{n-1}}{(t-z)^{n+m}}\Big]$$

since $(z^2-1)=(t^2-1)-(t-z)(t-z+2z)$.

$$=\Big[-\frac{(t^2-1)^n}{(t-z)^{n+m+1}}(m+3n+2)z+\frac{2n(t^2-1)^{n-1}}{(t-z)^{n+m}}$$

$$+\frac{(n+m+1)(t^2-1)^{n+1}}{(t-z)^{n+m+2}}-\frac{(n+m+1)(t^2-1)^n}{(t-z)^{n+m}}\Big].$$

Now *assume* that I is identically equal to

$$\alpha\,\frac{d}{dt}\left\{\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+1}}\right\}+\beta.\frac{d}{dt}\left\{\frac{t\,(t^2-1)^n}{(t-z)^{n+m}}\right\}$$

where α and β are constants to be determined. This expression $=$

$$\alpha\left[-\frac{(n+m+1)(t^2-1)^{n+1}}{(t-z)^{n+m+2}}+\frac{2(n+1)(t^2-1)^n}{(t-z)^{n+m}}\right.$$
$$\left.+\frac{2(n+1)z(t^2-1)^n}{(t-z)^{n+m+1}}\right]$$
$$+\beta\left[\frac{(t^2-1)^n}{(t-z)^{n+m}}-\frac{(n+m)(t^2-1)^n}{(t-z)^{n+m}}-z\frac{(n+m)(t^2-1)^n}{(t-z)^{n+m+1}}\right.$$
$$\left.+\frac{2n(t^2-1)^n}{(t-z)^{n+m}}+\frac{2n(t^2-1)^{n-1}}{(t-z)^{n+m}}\right].$$

If this be identically equal to I, we have, by equating the coefficients, four equations, viz:—

(i) $-(n+m+1)\alpha=(n+m+1)$

(ii) $2n\beta \qquad = 2n$

(iii) $2(n+1)\alpha-\beta(n-m)=-(m+3n+2)$

(iv) $(n-m+1)\beta+2(n+1)\alpha=-(n+m+1)$

which are satisfied by $\alpha=-1$ and $\beta=1$

Thus, the assumption has been justified and therefore it is proved that $\phi(z)=0$

2 (*b*). *To prove*

$$(z^2-1)\frac{dQ_n{}^m(z)}{dz}=nzQ_n{}^m(z)-(n+m)Q_{n-1}{}^m(z)$$

Proof:—The same as in 2 (*a*) except with the necessary difference in the contour and some change in the constant term multiplying the integrand.

3 (*a*). *To prove*

$$(z^2-1)\frac{dP_n{}^m(z)}{dz}-(n-m+1)P_{n+1}{}^m(z)+(n+1)zP_n{}^m(z)=0$$

Proof:—This can be obtained from 2 (*a*) by changing n into $-n-1$ but as another illustration we prove it by the method used above.

Denoting the left hand side of 3(*a*) by $\phi(z)$ we have

$$\phi(z)=\frac{e^{-n\pi i}}{4\pi\sin n\pi}\cdot(z^2-1)^{\frac{m}{2}}\cdot\frac{\Pi(n+m+1)}{\Pi(n)}$$

$$\cdot \frac{1}{2^n}\int^{(z+,\,1+,\,z-,\,1-)} I\,dt$$

where

$$I=\left[\frac{(z^2-1)\,(t^2-1)^n}{(t-z)^{n+m+2}}-\frac{z\,(t^2-1)^n}{(t-z)^{n+m+1}}-\frac{n-m+1}{2(n+1)}\cdot\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+2}}\right]$$

$$=\left[\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+2}}-\frac{(n-m+1)}{2(n+1)}\cdot\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+1}}\right.$$

$$\left.-\frac{z(t^2-1)^n}{(t-z)^{n+m+1}}-\frac{(t^2-1)^n}{(t-z)^{n+m}}\right].$$

Assume the expression in the rectangular brackets to be

$$=a\cdot\frac{d}{dt}\left\{\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+1}}\right\}$$ where a is a constant to be determined.

This expression $=$

$$\frac{2(n+1)\,a\,(t^2-1)^n}{(t-z)^{n+m}}+\frac{2(n+1)a\,z(t^2-1)^n}{(t-z)^{n+m+1}}$$

$$-\frac{a(n+m+1)\,(t^2-1)^{n+1}}{(t-z)^{n+m+2}}$$

If this be identically equal to I then by the comparison of coefficients we have

$$\left.\begin{array}{ll}\text{(i)} & 2(n+1)a=-1\\ \text{(ii)} & 2(n+1)a=-1\\ \text{(iii)} & 1-\dfrac{n-m+1}{2(n+1)}=-(m+n+1)a\end{array}\right\}\begin{array}{l}\text{which equations are}\\ \text{satisfied by}\\ a=-\dfrac{1}{2(n+1)}.\end{array}$$

Hence our assumption is justified and $\phi\,(z)$ is proved to be equal to zero.

3 (*b*). *To prove*

$$(z^2-1)\frac{dQ_n{}^m(z)}{dz}-(n-m+1)\,Q_{n+1}{}^m(z)+(n+1)\,zQ_n{}^m(z)=0$$

Proof :—Either change n into $-n-1$ in 2 (*b*) or use the method as in 3 (*a*).

§ 2

Before proceeding to generalize some of the recurrence formulae given by Heine it is to be remarked that the notations used in this article and the subsequent one for $P_n^m(z)$ and $Q_n^m(z)$ are those of Prof. Hobson.

I (*a*). *To prove*

$$\sqrt{(z^2-1)}\, P_{n+1}^{m+1}(z) = z\,(n-m+1)\, P_{n+1}^{m}(z) - (n+m+1)\, P_n^m(z)$$

Proof:—Let $\phi(z)$ denote

$$\sqrt{(z^2-1)}\, P_{n+1}^{m+1}(z) - z(n-m+1)\, P_{n+1}^{m}(z) + (n+m+1)\, P_n^m(z)$$

then $\phi(z)$

$$= \frac{e^{-n\pi i}}{4\pi \sin n\pi}\, \frac{1}{2^{n+1}}\, \frac{\Pi(n+m+1)}{\Pi(n+1)}\, (z^2-1)^{\frac{m}{2}} \times \int^{(z+,\ 1+,\ z-,\ 1-)} \mathrm{I}\,.\,dt$$

where

$$\mathrm{I} = \frac{(t^2-1)^{n+1}\,.\,(z^2-1)\,(n+m+2)}{(t-z)^{n+m+3}} - \frac{z(n-m+1)\,(t^2-1)^{n+1}}{(t-z)^{n+m+2}} + 2(n+1)\,\frac{(t^2-1)^n}{(t-z)^{n+m+1}}$$

$$= \frac{(t^2-1)^{n+2}\,(n+m+2)}{(t-z)^{n+m+3}} - \frac{z\,(t^2-1)^{n+1}}{(t-z)^{n+m+2}}(3n+m+5) + 2(n+1)\,\frac{(t^2-1)^n}{(t-z)^{n+m+1}} - \frac{(n+m+2)\,(t^2-1)^{n+1}}{(t-z)^{n+m+1}}\,.$$

Assume that I is identically equal to

$$\alpha\,\frac{d}{dt}\left[\frac{(t^2-1)^{n+2}}{(t-z)^{n+m+2}}\right] + \beta\,\frac{d}{dt}\left[\frac{t\,.\,(t^2-1)^{n+1}}{(t-z)^{n+m+1}}\right]$$

where α and β are constants to be determined.

The above expression

$$=\alpha\left[-\frac{(n+m+2)\,(t^2-1)^{n+2}}{(t-z)^{n+m+3}} + 2(n+2)\,\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+1}}\right.$$

$$+\frac{2(n+2)z\,(t^2-1)^{n+1}}{(t-z)^{n+m+2}}\Big]$$

$$+\beta\Big[(n-m+2)\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+1}}+\frac{2(n+1)^2\,(t-1)^n}{(t-z)^{n+m+1}}$$

$$-\frac{z(n+m+1)}{(t-z)^{n+m+2}}(t^2-1)^{n+1}\Big]$$

If this be identically equal to I, then by the comparison of coefficients we find

(i) $-(n+m+2)\,\alpha=n+m+2$

(ii) $(2n+1)\,\beta=(2n+1)$

(iii) $2(n+2)\,\alpha+(n-m+2)\,\beta=-(n+m+2)$

(iv) $2(n+2)\,\alpha-(n+m+1)\,\beta=-(3n+m+5)$.

which equations are satisfied by $\alpha=-1$, $\beta=1$.

Hence our assumption is justified and $\phi(z)$ is proved to be equal to zero.

1 (*b*) *To prove*

$$\sqrt{(z^2-1)}\,Q_{n+1}^{m+1}(z)=(n-m+1)z\,Q_{n+1}^{m}(z)-(n+m+1)\,Q_n^{m}(z)$$

Proof:—The same as in 1 (*a*) with the necessary changes.

2 (*a*) *To prove*

$$\frac{dP_n^m(z)}{dz}=\frac{1}{\sqrt{(z^2-1)}}P_n^{m+1}(z)+\frac{mz}{(z^2-1)}P_n^m(z)$$

Proof:—

Let $\phi(z)=(z^2-1)\dfrac{d\,P_n^m(z)}{dz}-\sqrt{z^2-1}\;P_n^{m+1}(z)-mz\,P_n^m(z)$

Then $\phi(z)=$

$$\frac{e^{-n\pi i}}{4\pi\sin n\pi}\,\frac{\Pi(n+m)}{\Pi(n)}\cdot\frac{1}{2^n}(z^2-1)^{\frac{m}{2}}\Big[\int^{(z+,\,1+,\,z-,\,1-)}\frac{(t^2-1)^n\cdot mz\cdot dt}{(t-z)^{n+m+1}}$$

$$+\int^{(z+,\,1+,\,z-,\,1-)}\frac{(n+m+1)\,(t^2-1)^n\,(z^2-1)}{(t-z)^{n+m+2}}\,dt\Big]$$

$$-\frac{e^{-n\pi i}}{4\pi\sin n\pi}\,\frac{\Pi(n+m)}{\Pi(n)}\cdot\frac{1}{2^n}\times$$

$$(z^2-1)^{\frac{m}{2}}\left[\int^{(z+,\,1+,\,z-,\,1-)} \frac{(z^2-1)\,(n+m+1)\,(t^2-1)^n}{(t-z)^{n+m+2}}\,dt\right.$$

$$\left.+\int^{(z+,\,1+,\,z-,\,1-)} \frac{mz\,.\,(t^2-1)^n}{(t-z)^{n+m+1}}\,dt\right].$$

$=0$, as the first member is equal, and opposite in sign, to the second.

2 (*b*). *To prove*

$$\frac{dQ_n^m(z)}{dz}=\frac{1}{\sqrt{(z^2-1)}}\,Q_n^{m+1}(z)+\frac{mz}{(z^2-1)}\,Q_n^m(z)$$

Proof:—Similar to that in 2 (*a*).

3 (*a*) *To prove*

$$\frac{dP_n^m(z)}{dz}=\frac{(n-m+1)\,(n+m)}{\sqrt{(z^2-1)}}\,P_n^{m-1}(z)-\frac{mz}{(z^2-1)}P_n^m(z)$$

Proof:—Let $\phi(z)=(z^2-1)\,\dfrac{dP_n^m(z)}{dz}-(n+m)\,(n-m+1)\times$

$$\sqrt{(z^2-1)}\,P_n^{m-1}(z)+mz\,P_n^m(z).$$

Then $\phi(z)=\dfrac{e^{-n\pi i}}{4\pi\sin n\pi}\;\dfrac{\Pi(n+m)}{\Pi(n)}\cdot\dfrac{1}{2^n}\,(z^2-1)^{-\frac{m}{2}}\times$

$$\int^{(z+,\,1+,\,z-,\,1-)} I\,dt$$

where

$$I=\left[\frac{mz\,(t^2-1)^n}{(t-z)^{n+m+1}}+\frac{(n+m+1)\,(t^2-1)^n\,(z^2-1)}{(t-z)^{n+m+2}}\right.$$

$$\left.+\frac{mz\,(t^2-1)^n}{(t-z)^{n+m+1}}-(n-m+1)\,\frac{(t^2-1)^n}{(t-z)^{n+m}}\right]$$

$$=\left[\frac{2\,mz\,(t^2-1)^n}{(t-z)^{n+m+1}}-(n-m+1)\,\frac{(t^2-1)^n}{(t-z)^{n+m}}+(n+m+1)\times\right.$$

$$\left.\left\{\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+2}}-2z\,\frac{(t^2-1)^n}{(t-z)^{n+m+1}}-\frac{(t^2-1)^n}{(t-z)^{n+m}}\right\}\right]$$

$$=\Big[\,2\,mz\,\frac{(t^2-1)^n}{(t-z)^{n+m+1}}-2\,(n+1)\,\frac{(t^2-1)^n}{(t-z)^{n+m}}$$

$$+\,(n+m+1)\,\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+2}}-\frac{2\,z\,(n+m+1)\,(t^2-1)^n}{(t-z)^{n+m+1}}\Big]$$

$$=\Big[\,(n+m+1)\,\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+2}}-\frac{2\,(n+1)\,z\,(t^2-1)^n}{(t-z)^{n+m+1}}$$

$$-\;2\,(n+1)\,\frac{(t^2-1)^n}{(t-z)^{n+m}}\Big]$$

$$=-\frac{d}{dt}\left\{\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+1}}\right\}\quad\text{as it is easy to see.}$$

$$\therefore\ \phi(z)=-\frac{e^{-\pi ni}}{4\pi\sin n\pi}\cdot\frac{\Pi(n+m)}{\Pi(n)}\frac{1}{2^n}(z^2-1)^{\frac{m}{2}}\times$$

$$\int^{(z+,\,1+,\,z-,\,1-)}\frac{d}{dt}\left\{\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+1}}\right\}dt$$

$$=0$$

3 (*b*). *To prove*

$$\frac{dQ_n^{\,m}(z)}{dz}=\frac{(n-m+1)\,(n+m)}{\sqrt{(z^2-1)}}\,Q_n^{\,m-1}(z)-\frac{mz}{z^2-1}\,Q_n^{\,m}(z).$$

Proof:—Similar to that in 3 (*a*) above.

4 (*a*). *To prove*

$$2\sqrt{(z^2-1)}\,\frac{dP_n^{\,m}(z)}{dz}=P_n^{\,m+1}(z)+(n-m+1)\,(n+m)\,P_n^{\,m-1}(z)$$

Proof :—Add 2 (*a*) and 3 (*a*) and we get the result.

4 (*b*) . Similarly by adding 2 (*b*) and 3 (*b*) we have

$$2\sqrt{(z^2-1)}\,\frac{dQ_n^{\,m}(z)}{dz}=Q_n^{\,m+1}(z)+(n-m+1)\,(n+m)\,Q_n^{\,m-1}(z)$$

5 (*a*). From 2 (*a*) and 3 (*a*) by subtraction we have

$$\frac{2\,mz}{\sqrt{(z^2-1)}}\,P_n^{\,m}(z)=(n-m+1)\,(n+m)\,P_n^{\,m-1}(z)-P_n^{\,m+1}(z)$$

5 (*b*). Similarly by subtraction from 2 (*b*) of 3 (*b*) we have

$$\frac{2\,mz\,Q_n^m(z)}{\sqrt{(z^2-1)}}=(n-m+1)(n+m)\,Q_n^{m-1}(z)-Q_n^{m+1}(z)$$

6 (*a*) Multiplying 4 (*a*) by $\frac{2z}{\sqrt{(z^2-1)}}$ and using 5 (*a*) we get

$$4z\,.\,\frac{d\,P_n^m(z)}{dz}=-\frac{2n(n+1)}{m^2-1}P_n^m(z)-\frac{P_n^{m+2}(z)}{m+1}$$

$$+\frac{(n+m)(m+n-1)(n-m+1)(n-m+2)}{m-1}P_n^{m-2}(z)$$

6 (*b*). Similarly Multiplying 4 (*b*) by $\frac{2z}{\sqrt{(z^2-1)}}$ and using 5 (*b*) we get

$$4z\,\frac{dQ_n^m(z)}{dz}=-\frac{2n(n+1)}{m^2-1}Q_n^m(z)-\frac{Q_n^{m+2}(z)}{m+1}$$

$$+\frac{(n+m)(n+m-1)(n-m+1)(n-m+2)}{m-1}Q_n^{m-2}(z)$$

7 (*a*).* From 6 (*a*), by substituting for $\frac{z\,dP_n^m(z)}{dz}$ in terms of $\frac{z^2\,P_n^m(z)}{z^2-1}$ and $\frac{z}{\sqrt{(z^2-1)}}P_n^{m+1}(z)$ by means of 2 (*a*) and using 5 (*a*) we have

$$\frac{4\,m\,z^2}{z^2-1}P_n^m(z)=\frac{2\,m\,(m^2-1-n^2-n)}{m^2-1}P_n^m(z)+\frac{1}{m+1}P_n^{m+2}(z)$$

$$+\frac{(n+m)(n+m-1)(n-m+1)(n-m+2)}{m-1}P_n^{m-2}(z)$$

7 (*b*). From 6 (*b*) by substituting for $z\,\frac{dQ_n^m(z)}{dz}$ in terms of $\frac{z^2Q_n^m(z)}{z^2-1}$ and $\frac{z}{\sqrt{(z^2-1)}}$ $Q_n^{m+1}(z)$ by means of 2 (*b*) and using 5 (*b*) we have

$$\frac{4\,m\,z^2}{z^2-1}Q_n^m(z)=\frac{2m(m^2-1-n^2-n)}{m^2-1}Q_n^m(z)+\frac{1}{m+1}Q_n^{m+2}(z)$$

$$+\frac{(n+m)(n+m-1)(n-m+1)(n-m+2)}{m-1}Q_n^{m-2}(z).$$

* The corresponding formula given by Heine (*l. c.*) as formula (*c*) on p. 258 seems to be wrong.

§ 3

1 (*a*). *To prove*

$$-\sqrt{(z^2-1)}\,P_n^{m+1}(z)-(2m+n+1)\,zP_n^m(z)$$
$$-m(m+n)\sqrt{(z^2-1)}\,P_n^{m-1}(z)+(n+1)\,P_{n+1}^m(z)=0.$$

Proof :—Let the left hand side of 1 (*a*) be denoted by $\phi(z)$.

Then $\phi(z)=$

$$\frac{e^{-n\pi i}}{4\pi\sin n\pi}\cdot\frac{\Pi(n+m)}{\Pi(n)}\,\frac{1}{2^{n+1}}\,(z^2-1)^{\frac{m}{2}}\left[\int^{(z+,\,1+,\,z-,\,1-)} \mathrm{I}\,dt\right]$$

where

$$\mathrm{I}=\Bigg[-2\,(n+m+1)\left\{\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+2}}-\frac{(t^2-1)^n}{(t-z)^{n+m}}\right.$$
$$\left.-2z\,\frac{(t^2-1)^n}{(t-z)^{n+m+1}}\right\}-2(2m+n+1)z\,\frac{(t^2-1)^n}{(t-z)^{n+m+1}}$$
$$-2m\,\frac{(t^2-1)^n}{(t-z)^{n+m}}+(n+m+1)\,\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+2}}\Bigg]$$
$$=\Bigg[-(n+m+1)\,\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+2}}+2(n+1)\,\frac{(t^2-1)^n}{(t-z)^{n+m}}$$
$$+2(n+1)\,z\,\frac{(t^2-1)^n}{(t-z)^{n+m+1}}\Bigg]$$
$$=\frac{d}{dt}\left\{\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+1}}\right\}$$

$$\therefore\ \phi(z)=0$$

1 (*b*). *To prove*

$$-\sqrt{(z^2-1)}\,Q_n^{m+1}(z)-(2m+n+1)z\,Q_n^m(z)$$
$$-m(m+n)\sqrt{(z^2-1)}\,Q_n^{m-1}(z)+(n+1)\,Q_{n+1}^m(z)=0$$

Proof :—Similar to that in 1 (*a*) with necessary changes.

Changing n into $-n-1$ in 1 (*a*) and 1 (*b*) we have

2 (*a*). $$\sqrt{(z^2-1)}\,P_n^{m+1}(z)+(2m-n)z\,P_n^m(z)$$
$$+m(m-n-1)\sqrt{(z^2-1)}\,P_n^{m-1}(z)+n\,P_{n-1}^m(z)=0.$$

2 (*b*). $\sqrt{(z^2-1)}\,Q_n^{m+1}(z) + (2m-n)z\,Q_n^m(z)$

$+m(m-n-1)\,\sqrt{(z^2-1)}\,Q_n^{m-1}(z) + n\,Q_{n-1}^m(z)=0.$

3 (*a*). *To prove*

$(n+m)\,\sqrt{(z^2-1)}\,P_n^{m-1}(z)+z\,P_n^m(z)-P_{n+1}^m(z)=0.$

Proof:—The left hand side of 3 (*a*)

$$=\frac{e^{-n\pi i}}{4\pi\sin n\pi}\;\frac{\Pi(n+m)}{\Pi(n)}\;\frac{1}{2^n}\;(z^2-1)^{\frac{m}{2}}\left[\int^{(z+,\,1+,\,z-,\,1-,)} \mathrm{I}.\;dt\right]$$

Where

$$\mathrm{I}\;=$$

$$=\frac{(t^2-1)^n}{(t-z)^{n+m}}+z\,\frac{(t^2-1)^n}{(t-z)^{n+m+1}}-\frac{(n+m+1)}{2(n+1)}\cdot\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+2}}$$

$$=\frac{1}{2(n+1)}\left[-(n+m+1)\,\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+2}}+\frac{2(n+1)\,z(t^2-1)^n}{(t-z)^{n+m+1}}+\frac{2(n+1)\,(t^2-1)^n}{(t-z)^{n+m}}\right]$$

$$=\frac{1}{2(n+1)}\,\frac{d}{dt}\left\{\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+1}}\right\}$$

$$\therefore\quad \phi(z)=\frac{e^{-n\pi i}}{4\pi\sin n\pi}\;\frac{\Pi(n+m)}{\Pi(n+1)}\cdot\frac{1}{2^{n+1}}\,(z^2-1)^{\frac{m}{2}}\times$$

$$\left[\int^{(z+,\,1+,\,z-,\,1-)}\frac{d}{dt}\left\{\frac{(t^2-1)^{n+1}}{(t-z)^{n+m+1}}\right\}dt\right]$$

$$=0$$

3 (*b*). *To prove*

$(n+m)\,\sqrt{(z^2-1)}\,Q_n^{m-1}(z)+z\,Q_n^m(z)-Q^m_{n+1}(z)=0.$

Proof:—Similar to that in 3 (*a*) with the necessary changes.

Changing n into $-n-1$ in 3 (*a*) and 3 (*b*) we have

4 (*a*). $(n-m+1)\,\sqrt{(z^2-1)}\,P_n^{m-1}(z)-z\,P_n^m(z)+P^m_{n-1}(z)=0.$

and

4 (b). $(n-m+1)\sqrt{(z^2-1)}\,Q_n^{m-1}(z)-z\,Q_n^m(z)+Q^m{}_{n-1}(z)=0.$

Combining 1 (a) with 2 (a) and 1 (b) with 2(b) we get

$$5\ (a).\quad (2n+1)z\,P_n^m(z)+m\,(2n+1)\sqrt{(z^2-1)}\,P_n^{m-1}(z)$$
$$-n\,P^m{}_{n-1}(z)-(n+1)\,P^m{}_{n+1}(z)=0.$$

and

$$5\ (b).\quad (2n+1)z\,Q_n^m(z)+m\,(2n+1)\sqrt{(z^2-1)}\,Q_n^{m-1}(z)$$
$$-n\,Q^m{}_{n-1}(z)-(n+1)\,Q^m{}_{n+1}(z)=0.$$

Similarly combining 3 (a) with 4 (a) and 3 (b) with 4 (b) we have

$$6\ (a).\ (2n+1)\sqrt{(z^2-1)}\,P_n^{m-1}(z)+P_{n-1}{}^m(z)-P_{n+1}{}^m(z)=0.$$

and

$$6\ (b).\ (2n+1)\sqrt{(z^2-1)}\,Q_n^{m-1}(z)+Q_{n-1}{}^m(z)-Q_{n+1}{}^m(z)=0.$$

Combining 5 (a) with 6 (a) and 5 (b) with 6 (b) we get

$$7\ (a).\ (2n+1)z\,P_n^m(z)+(m-1)\,(2n+1)\sqrt{(z^2-1)}\,P_n^{m-1}(z)$$
$$-(n+1)\,P_{n-1}{}^m(z)-n\,P_{n+1}{}^m(z)=0.$$

and

$$7\ (b).\ (2n+1)z\,Q_n^m(z)+(m-1)\,(2n+1)\sqrt{(z^2-1)}\,Q_n^{m-1}(z)$$
$$-(n+1)\,Q_{n-1}{}^m(z)-nQ_{n+1}{}^m(z)=0.$$

Multiplying 1 (a) and 1 (b) by n and 2 (a) and 2 (b) by $(n+1)$ and then subtracting 2 (a) and 2 (b) from 1 (a) and 1 (b) respectively we get

8 (a).

$$n(n+1)\left\{P_{n+1}{}^m(z)-P_{n-1}{}^m(z)\right\}-2(n+1)\left\{\sqrt{(z^2-1)}P_n^{m+1}(z)\right.$$
$$\left.+2mz\,P_n^m(z)+m(m-1)\sqrt{(z^2-1)}\,P_n^{m-1}(z)\right\}=0$$

and

8 (b).

$$n(n+1)\left\{Q^m{}_{n+1}(z)-Q^m{}_{n-1}(z)\right\}-2(n+1)\left\{\sqrt{(z^2-1)}\,Q_n^{m+1}(z)\right.$$
$$\left.+2mz\,Q_n^m(z)+m(m-1)\sqrt{(z^2-1)}\,Q_n^{m-1}(z)\right\}=0.$$

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Note

For the proofs of 3 (*b*) in § 1 and 2 (*b*) and 4 (*b*) in § 3 by changing n into $-n-1$, it should be noted that, in spite of the relation

$$P_n^m = \frac{e^{-m\pi i}}{4\pi \cos n\pi} \left\{ Q_n^m . \sin (n+m)\pi - Q_{-n-1}^m . \sin (n-m)\pi \right\}$$

(*Hobson's paper*, *l. c.*, p. 462)

being not so simple as

$$P_n^m = P_{-n-1}^m,$$

these recurrence formulae still prove to be true on account of the corresponding recurrence formulae for P_n^m, as may be seen by actual substitution.

Paper II

ON CERTAIN EXPANSIONS OF ZERO IN SERIES OF ASSOCIATED LEGENDRE FUNCTIONS, $P_n^m(\mu)$

By

N. G. SHABDE.

Introduction:—The object of the present paper is to show how zero can be expanded in series of $P_n^m(\mu)$, n and m being integers and $\mu = \cos\theta$. I give a number of expansions of zero in terms of $P_n^m(\mu)$; the methods used being (i) a method similar to that used by Lindemann for expansions of zero in Lamé's functions and (ii) a method different from that used by Lindemann.

Zero has been expanded in infinite series of Bessel's functions by Niels Nielsen*. Expansions of zero in series of Lamé's functions have been given by F. Lindemann†. Similar expansions of zero in series of $P_n(\mu)$ are not possible with integral n but may be possible with n non-integral, as in the latter case the functions do not always form an orthogonal system. Such non-orthogonal systems of Legendre's functions have been recently completely investigated by Ganesh Prasad ‡. Some of the series obtained by Ganesh Prasad § and the present

**Math. Annalen*, Bd. 52, pp. 582-587, "Sur le développement du zero en séries de fonctions cylindriques".

†*Math. Annalen*, Bd. 19, pp. 323-386, "Entwicklung der Functionen einer complexen Variablen nach Laméschen Functionen und nach Zugeordneten der Kugelfunctionen". See specially pp. 361-378 under the heading "Nullentwicklungen"; also p. 381.

‡*Proceedings of the Benares Mathematical Society*, Vol XII, "On the non-orthogonal systems of Legendre's functions", pp. 33-42.

§ *Bulletin of the Calcutta Mathematical Society*, Vol. XXIII. No. 3, pp. 116-124 "On the summation of infinite series of Legendre's functions", (second paper). See series II (p. 116). This series can be written as

$$0 = \frac{\pi}{4} K_0(\cosh\psi) - K_1(\cosh\psi) + \frac{1}{3} K_3(\cosh\psi) - \frac{1}{5} K_5(\cosh\psi)$$
$$+ - \text{ to infinity,}$$

where

$$0 < \psi < \frac{\pi}{2} \text{ and } K_p(\cosh\psi) = P_{-\frac{1}{2}+pi}(\cosh\psi)$$

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author * can be exhibited as series for zero in terms of $P_n(\mu)$, n non-integral. C. Fox† also gives a series for zero in terms of $P_n(\mu)$, n non-integral.

* *Bull. Calcutta Math. Society*, Vol. XXIII, No. 3, pp 155-182. "On the summation of infinite series of Legendre's functions" See series 1 and 2 (pp. 155-159). These can be written as

$$(1)\quad O=\frac{\pi}{2\sqrt{(2)}}\ K_o(\cosh\ \psi)-K_1(\cosh\ \psi)-\frac{1}{3}\ K_3(\cosh\ \psi)$$

$$+\frac{1}{5}\ K_5(\cosh\ \psi)-\ -+\ +\ \text{to infinity,}$$

where

$$0<\psi<\frac{\pi}{2}\ \text{and}\ K_p(\cosh\psi)=P_{-\frac{1}{2}+pi}(\cosh\psi)$$

as in the last foot-note

nd $$(2)\quad O=\frac{\pi}{2\sqrt{(3)}}\ K_o(\cosh\psi)-K_1(\cosh\psi)+\frac{1}{5}\ K_5(\cosh\psi)$$

$$-\frac{K_7(\cosh\psi)}{7}+\frac{K_{11}(\cosh\psi)}{11}-+\ldots\ \text{to infinity,}$$

where

$$0<\psi<\frac{\pi}{3}\ \text{and}\ K_p(\cosh\psi)=P_{-\frac{1}{2}+pi}(\cosh\psi).$$

† *Proceedings, London Mathematical Society*, Vol. XXIII 26, pp. 35-80, "Some further contribution to the theory of null series and their connexion with null integrals" Examples given are

$$(i)\quad \tfrac{1}{2}\ P_{-\frac{1}{2}}{}^{m}(\cos\phi)+\sum_{n=1}^{\infty}(-1)^n\ P_{nx-\frac{1}{2}}{}^{m}(\cos\phi)=0\ \text{if}$$

$$0<\phi<\pi;\ 0<x\phi<\pi\ \text{and}\ m<\tfrac{1}{2}$$

and

$$(ii)\quad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\ P_{x(n-\frac{1}{2})-\frac{1}{2}}(\cos\phi)=\tfrac{1}{4}\ \pi\ P_{-\frac{1}{2}}(\cos\phi)\ \text{whenever}$$

$0<\phi<\pi$ and $0\leq x<\pi$ (p. 76).

Methods:—Two methods can be followed for the required expansion.

I. One is analogous to that used by Lindemann for Lamé's-function-expansions of zero. It is to assume the required expansion in the form $\sum_{n=0}^{n} \sum_{m=0}^{n} A_n^m P_n^m(\mu)$, to expand $P_n^m(\mu)$ in series of μ and to equate to zero the coefficients of different powers of μ. The coefficients, A_n^m's, can be determined from the resulting equations. It is easy to see that if $S_n=0$ gives the expansion of the nth degree, the general expansion of zero of the nth degree will be $S_n+\lambda_1 S_{n-1}+\ldots=0$, where λ's are arbitrary constants.

II. The second method is a well-known one and is indicated in several books on Spherical Harmonics. It is to construct rational integral functions of $\cos\theta$, $\sin\theta$. $\cos\phi$, $\sin\theta\sin\phi$ and to expand them in series of $P_n^m(\cos\theta)$. We finally put $\phi=0$ or $\frac{\pi}{2}$ and get the required expansion for zero.

$$(1)\quad O=-P_o^o(\mu)+2\left[\frac{5}{4}P_2^2(\mu)+\frac{9\cdot2!}{6!}\cdot P_4^2(\mu)\right.$$
$$\left.+\frac{13\cdot4!}{8!}P_6^2(\mu)+\ldots\ldots\right].$$

Proof:—It is known* that

$$\cos 2\phi=2\cos 2\phi\left[\frac{5}{4}P_2^2(\mu)+\frac{9\cdot2!}{6!}.P_4^2(\mu)+\frac{13\cdot4!}{8!}P_6^2(\mu)+\ldots.\right]$$

Putting $\phi=0$ and considering that $1=P_o^o(\mu)$, we get the required expansion.

$$(2)\quad O=-2P_o^o(\mu)+2P_2^0(\mu)+P_2^2(\mu)$$

Proof:—The result is well-known and almost obvious. But we shall try to obtain it by using Lindemann's method.

Let $o=A_o^o P_o^o(\mu)+A_2^o P_2^o(\mu)+A_2^2 P_2^2(\mu)$ ....(1)

But $P^o_2(\mu)=\frac{3\mu^2-1}{2}$, $P^2_2(\mu)=3\ (1-\mu^2)$. Substituting these values in (1), and equating to zero the coefficient of μ^2 and the term independent of μ, we have the equations,

* Ganesh Prasad : *Spherical Harmonics. Part* I, p 119. Ex. 9

$$\left.\begin{array}{l} A^o_o - \frac{1}{2} A^o_2 + 3\ A^2_2 = 0 \\ \text{and } A^o_2 = 2A^2_2. \end{array}\right\}$$

These give $2\ A^2_2 = -A^o_0$ and $A^o_2 = 2A^2_2$ and the result follows immediately.

(3) $O = 6\ P^o_1(\mu) - 6\ P^o_3(\mu) - P^2_3(\mu).$

Proof: — Assume

$$O = A^2_3\ P^2_3(\mu) + A^o_3\ P^o_3(\mu) + A^o_1\ P^o_1(\mu).$$

But

$$P^2_3(\mu) = 15\ (\mu - \mu^3),$$

$$P^o_3(\mu) = \frac{5\ \mu^3 - 3\mu}{2}$$

and $$P^o_1(\mu) = \mu.$$

Equating to zero the coefficients of μ^3 and μ, we have,

$$\left.\begin{array}{l} -15\ A^2_3 + \frac{5}{2}\ A^o_3 = 0 \ldots\ldots\ldots\ldots\text{(i)} \\ A^o_1 + 15\ A^2_3 - \frac{3}{2}\ A^o_3 = 0 \ldots\ldots\ldots\text{(ii)} \end{array}\right\}$$

These give $\quad A^2_3 = \frac{1}{6}\ A^o_3 = -\frac{1}{6}\ A^o_1$.

Hence

$6P^o_1(\mu) - 6\ P^o_3(\mu) - P^2_3(\mu) = 0$, taking A^o_1 different from zero.

In general, the expansion of zero in $P^m_n(\mu)$ of the third degree is

$$P^2_3(\mu) + 6\ P^o_3(\mu) - 6P^o_1(\mu) + \lambda\{P^2_2(\mu) + 2P^o_2(\mu) - 2P^o_o(\mu)\} = 0,$$

where λ is an arbitrary constant.

(4) $$O = \left[\frac{1}{3}\left\{2\ P^o_2 + P^o_o\right\} - \frac{2\cdot 4}{5\cdot 7\cdot 9}\left\{9\ P^o_4 + 5\cdot\frac{9}{2}\ P^o_2 + \frac{9\cdot 7}{2\cdot 4}\ P^o_o\right\}\right] - \frac{1}{12}\cdot\frac{2\cdot 4}{5\cdot 7\cdot 9}\cdot\left[9\cdot P^2_4 + \frac{5\cdot 9}{2}\cdot P^2_2\right]$$

Proof:—

Putting $\phi = 0$ in the expansion

$$\mu^2 (1-\mu^2) \sin^2\phi = \tfrac{1}{2}\mu^2 (1-\mu^2)(1-\cos 2\phi)$$

$$= \frac{1}{2}\left[\frac{1}{3}\{2P_2+P_0\} - \frac{2\cdot 4}{5\cdot 7\cdot 9}\left\{9\,P_4 + \frac{5\cdot 9}{2}.\,P_2 + \frac{9\cdot 7}{2\cdot 4}P_0\right\}\right]$$

$$-\frac{1}{2}\cos 2\phi\left[\frac{1}{12}.\ \frac{2\cdot 4}{5\cdot 7\cdot 9}\left\{9\,P^2{}_4 + \frac{5\cdot 9}{2}\,P^2{}_2\right\}\right],$$

we have the required result.

(5)

$$O = \frac{1}{6930}\,P^3{}_6\,(\mu) + \frac{1}{1540}\,P^3{}_4\,(\mu) + \frac{2}{693}\,P^1{}_6\,(\mu)$$

$$-\frac{1}{770}P^1{}_4\,(\mu) - \frac{1}{63}\,P^1{}_2\,(\mu).$$

Proof:—It is known* that

$$\cos^3\theta\cdot\ \sin^3\theta\cdot\sin\phi\cdot\cos^2\phi$$

$$= \left\{\frac{1}{6930}\,P^3{}_6\,(\mu) + \frac{1}{1540}\,P^3{}_4\,(\mu)\right\}\sin 3\phi$$

$$-\left\{\frac{2}{693}\,P^1{}_6\,(\mu) - \frac{1}{770}\,P^1{}_4\,(\mu) - \frac{1}{63}P^1{}_2\,(\mu)\right\}\sin\phi.$$

Put $\phi = \frac{\pi}{2}$ and we have the required result .

(6)

$$O = \frac{1}{15}.\ P^4{}_8\,(\mu) + \frac{13}{30}P^4{}_6\,(\mu) + \frac{9}{8}\,P^4{}_4\,(\mu) + 2P^2{}_8\,(\mu)$$

$$-\frac{99}{4}P^2{}_4\,(\mu) - \frac{195}{4}\,P^2{}_2\,(\mu)$$

Proof:—

$$\cos^4\theta\cdot\sin^4\theta\cdot\ \sin^2\phi\cdot\ \cos^2\phi =$$

$$\frac{1}{8}\cdot\cos^4\theta\cdot\ \sin^4\theta\cdot\ (1-\cos 4\phi).$$

* Ferrers : *Spherical Harmonics*, p. 92.

Now

$$(i) -\frac{\cos 4\phi}{8} \cdot \cos^4\theta \cdot \sin^4\theta = -\frac{\cos 4\phi}{8} \cdot \frac{(1-\mu^2)^{\frac{4}{2}}}{8\cdot7\cdot6\cdot5} \frac{d^4}{d\mu^4}(\mu^8)$$

$$= -\frac{1\cdot2\cdot3\cdot4\cdots8\cdot\cos 4\phi}{3\cdot5\cdot7\cdot9\cdots17\cdot8\cdot7\cdot6\cdot5}\Big[17\, P^4_8 + 13\cdot\frac{17}{2}\, P^4_6$$

$$+ 9\cdot\frac{17\cdot15}{2\cdot4}\, P^4_4\Big].$$

Similarly,

$$(ii) \quad \frac{1}{8}\cos^4\theta\cdot\sin^4\theta =$$

$$\frac{1}{8}\Big[\frac{4}{6\cdot5}\cdot\frac{6!}{3\cdot5\cdots13}\Big\{13\, P^2_6 + 9\cdot\frac{13}{2}\cdot P^2_4 + 5\cdot\frac{13\cdot11}{2\cdot4}\, P^2_2\Big\}$$

$$-\frac{1}{8\cdot7}\cdot\frac{1\cdot2\cdots8}{3\cdot5\cdots17}\Big\{17\, P^2_8 + 13\cdot\frac{17}{2}\, P^2_6 + 9\cdot\frac{17\cdot15}{2\cdot4}\, P^2_4$$

$$+ 5\cdot\frac{17\cdot15\cdot13}{2\cdot4\cdot6}\cdot P^2_2\Big\}\Big].$$

From (i) and (ii) we have by putting $\phi = 0$ the required result.

In general, if the right side of (6) be denoted by S_8 and that of (5) by S_6, we have $S_8 + \lambda S_6 = 0$, λ being an arbitrary constant.

(7)

$$O = -\frac{10!}{10\cdot9\cdot8\cdot7\cdot6\cdot16\cdot3\cdot5\cdots21}\Big\{21\, P^5_{10} + 17\cdot\frac{21}{2}\, P^5_8$$

$$+ 13\cdot\frac{21\cdot19}{2\cdot4}\, P^5_6\Big\}$$

$$-\frac{8!}{16\cdot8\cdot7\cdot6\cdot3\cdot5\cdots17}\Big\{17\cdot P^3_8 + 13\cdot\frac{17}{2}\, P^3_6$$

$$+ 9\cdot\frac{17\cdot15}{2\cdot4}\, P^3_4\Big\}$$

$$+\frac{10!}{16\cdot10\cdot9\cdot8\cdot3\cdot5\cdots21}\Big\{21\, P^3_{10} + 17\cdot\frac{21}{2}\, P^3_8 + 13\cdot\frac{21\cdot19}{2\cdot4}\, P^3_6$$

$$+9\cdot\frac{21\cdot19\cdot17}{2\cdot4\cdot6}P^3_4\Big\}$$

$$+\frac{2}{16}\cdot\frac{6!}{6\cdot3\cdot5\cdots13}\Big\{13\,P^1_6+9\cdot\frac{13}{2}P^1_4+5\,\frac{13\cdot11}{2\cdot4}P^1_2\Big\}$$

$$-\frac{2}{16}\cdot\frac{2}{8}\frac{8!}{3\cdot5\,..17}\Big\{17\,P^1_8+13.\,\frac{17}{2}.\,P^1_6+9.\,\frac{17\cdot15}{2\cdot4}P^1_4$$

$$+5.\,\frac{17\cdot15\cdot13}{2\cdot4\cdot6}P^1_2\Big\}$$

$$+\frac{2}{16}\cdot\frac{1}{10}\cdot\frac{10!}{3\cdot5\,..21}\Big\{21\,P^1_{10}+17.\,\frac{21}{2}\cdot P^1_8+13.\,\frac{21\cdot19}{2\cdot4}.P^1_6$$

$$+9.\,\frac{21\cdot19\cdot17}{2\cdot4\cdot6}.\,P^1_4+5\,\frac{21\cdot19\cdot17\cdot15}{2\cdot4\cdot6\cdot8}P^1_2\Big\}.$$

Proof:—

$$\mu^5(1-\mu^2)^{\frac{5}{2}}.\sin^3\phi.\cos^2\phi$$

$$=\mu^5(1-\mu^5)^{\frac{5}{2}}\frac{1}{10}\Big\{2\sin\phi-\sin5\phi+\sin3\phi\Big\}......(i)$$

Now

$$(a)\quad -\sin5\phi.\,\frac{1}{16}(1-\mu^2)^{\frac{5}{2}}\frac{1}{10\cdot9\cdot8\cdot7\cdot6}\frac{d^5}{d\mu^5}\mu^{10}$$

$$=-\sin5\phi.\,\frac{1}{16}(1-\mu^2)^{\frac{5}{2}}\frac{1}{10\cdot9\cdot8\cdot7\cdot6}\frac{d^5}{d\mu^5}\frac{10!}{3\cdot5...21}\Big\{21.P_{10}(\mu)+17.\frac{21}{2}P_8(\mu)$$

$$+13.\,\frac{21\cdot19}{2\cdot4}P_6(\mu)$$

$$+9.\,\frac{21\cdot19\cdot17}{2\cdot4\cdot6}.\,P_4(\mu)+5.\,\frac{21\cdot19\cdot17\cdot15}{2\cdot4\cdot6\cdot8}P_2+\frac{21\cdot19\cdot17\cdot15\cdot13}{2\cdot4\cdot6\,8\cdot10}\Big\}$$

$$=-\sin5\phi.\,\frac{1}{16.}\,\frac{1}{10\cdot9\cdot8\cdot7\cdot6}\Big\{\frac{10!}{3\cdot5...21}\Big\}\Big\{21P_{10}^5+17.\,\frac{21}{2}P_8^5$$

$$+13.\,\frac{21\cdot19}{2\cdot4}P^5_6\Big\}.$$

$$(b)\quad \frac{1}{16}\sin3\phi.\,\mu^5.(1-\mu^2)\,(1-\mu^2)^{\frac{3}{2}}=\frac{1}{16}(1-\mu^2)^{\frac{3}{2}}\Big\{\mu^5-\mu^7\Big\}$$

$$\times\sin3\phi=\frac{1}{16}\sin3\phi.\,(1-\mu^2)^{\frac{3}{2}}\frac{1}{8\cdot7\cdot6}\frac{d^3}{d\mu^3}(\mu^8)$$

$$-\frac{1}{16}\sin 3\phi.\ (1-\mu^2)^{\frac{3}{2}}\frac{1}{10\cdot9\cdot8}\frac{d^3}{d\mu^3}(\mu^{10})$$

$$=\frac{\sin 3\phi.\ 8!}{3\cdot5\ldots17\cdot16\cdot8\cdot7\cdot6}\left\{17\ P_8{}^3+13.\ \frac{17}{2}\ P_6{}^3+9.\ \frac{17.15}{2\cdot4}\ P_4{}^3\right\}$$

$$-\frac{\sin 3\phi.\ 10!}{16\cdot10\cdot9\cdot8\cdot3\cdot5\ldots21}\left\{21\ P_{10}{}^3+17.\frac{21}{2}\ P_8{}^3+13.\ \frac{21\cdot19}{2\cdot4}\ P_6{}^3\right.$$

$$\left.+9.\ \frac{21\cdot19\cdot17}{2\cdot4\cdot6}.P_4{}^3\right\}.$$

(c) $$\frac{1}{16}\mu^5(1-\mu^2)^2(1-\mu^2)^{\frac{1}{2}}.\,2\ \sin\phi=\frac{2\ \sin\phi}{16}\left(1-\mu^2\right)^{\frac{1}{2}}$$

$$\times\left\{\mu^5-2\ \mu^7+\mu^9\right\}$$

$$=\frac{2\ \sin\phi}{16}(1-\mu^2)^{\frac{1}{2}}\left\{\frac{1}{6}\ \frac{d}{d\mu}\mu^6-\frac{2}{8}\ \frac{d}{d\mu}\ \mu^8+\frac{1}{10}\ \frac{d}{d\mu}\ \mu^{10}\right\}$$

$$=\frac{2\ \sin\phi}{16}.\ \frac{1}{6}\left[\frac{6!}{3\cdot5\ldots13}\left\{13.\ P^1{}_6+\frac{9\cdot13}{2}.\ P^1{}_4+5.\ \frac{13\cdot11}{2\cdot4}\ P^1{}_2\right\}\right]$$

$$-\frac{2}{16}.\frac{2}{8}.\ \sin\phi\ \left[\frac{8!}{3\cdot5\ldots17}\left\{17P^1{}_8+13.\ \frac{17}{2}\ P^1{}_6+9.\ \frac{17\cdot15}{2\cdot4}.\ P^1{}_4\right.\right.$$

$$\left.\left.+5.\ \frac{17\cdot15\cdot13}{2\cdot4\cdot6}\ P^1{}_2\right\}\right]+\frac{2\ \sin\phi}{16}.\ \frac{1}{10}\left[\frac{10!}{3\cdot5\ldots21}\left\{21\ P^1{}_{10}+17.\frac{21}{2}\ P^1{}_8\right.\right.$$

$$\left.\left.+13.\ \frac{21\cdot19}{2\cdot4}\ P^1{}_6+9.\ \frac{21\cdot19\cdot17}{2\cdot4\cdot6}\ P^1{}_4+5.\ \frac{21\cdot19\cdot17\cdot15}{2\cdot4\cdot6\cdot8}\ P^1{}_2\right\}\right].$$

Thus (i)$=(a)+(b)+(c)$. Putting $\phi=\frac{\pi}{2}$ on both the sides, we have the required expansion.

(8)

$$\cos^8\theta.\ \sin^8\theta.\ \sin^4\phi.\ \cos^4\phi$$

$$=\mu^8\,(1-\mu^2)^{\frac{8}{2}}\,\frac{1}{2^7}\left\{\cos 8\,\phi-4\ \cos4\phi+6\right\}.$$

I. $$(1-\mu^2)^{\frac{8}{2}}\ \frac{1}{9\cdot10\cdot11\cdots16}\ \frac{d^8}{d\mu^8}\mu^{16}$$

$$=\ (1-\mu^2)^{\frac{8}{2}}\frac{1}{9\cdot10\ldots16}\ .\frac{1\cdot2\cdots16}{3\cdot5\cdot7\cdots33}\ \times$$

$$\frac{d^8}{d\mu^8}\left\{33\cdot P_{16}(\mu)\ +\frac{29\cdot33}{2}\ P_{14}(\mu)\right.$$

$$+25\ .\ \frac{33\cdot31}{2\cdot4}\ P_{12}\,(\mu)+21\ .\ \frac{33\cdot31\cdot29}{2\cdot4\cdot6}\ P_{10}(\mu)$$

$$+\frac{17\cdot 33\cdot 31\cdot 29\cdot 27}{2\cdot 4\cdot 6\cdot 8}\,.\ P_8(\mu)$$

$$+\frac{13\cdot 33\cdot 31\cdot 29\cdot 27\cdot 25}{2\cdot 4\cdot 6\cdot 8\cdot 10}\ P_6(\mu)+\frac{9\cdot 33\cdot 31\cdot 23}{2\cdot 4\cdot 12}\ P_4(\mu)$$

$$+5.\frac{33\cdot 31\cdot 21}{2\cdot 4..\ 14}\,P_2(\mu)+\frac{33\cdot 31\cdot 19}{2\cdot 4\ldots 16}\Big\}$$

$$=\frac{1\cdot 2\ldots 16}{(9\cdot 10\ldots 16)\ (3\cdot 5\cdots 33)}\Big[33\cdot P^8{}_{16}\ (\mu)+29\,.\frac{33}{2}\,.\ P^8{}_{14}$$

$$+\ 25\,.\,\frac{33\cdot 31}{2\cdot 4}\ P^8{}_{12}\ (\mu)+\ \frac{21\cdot 33\cdot 31\cdot 29}{2\cdot 4\cdot 6}\,.\ P^8{}_{10}\ (\mu)$$

$$+17\,.\,\frac{33\cdot 31\cdot 27}{2\cdot 4\cdot 6\cdot 8}\,.\ P^8{}_8\ (\mu)\Big]\,.$$

II. $(1-\mu^2)^{\frac{4}{2}}\ [\mu^8\ (1-\mu^2)^2]=(1-\mu^2)^{\frac{4}{2}}\ [\mu^8\ (1-2\mu^2+\mu^4)]$

$$=\ (1-\mu^2)^{\frac{4}{2}}\Big[\frac{1}{12\cdot 11\cdot 10\cdot 9}\,\frac{d^4}{d\mu^4}\,(\mu^{12})\ +\ \frac{1}{16\cdot 15\cdot 14\cdot 13}\,\frac{d^4}{d\mu^4}(\mu^{16})$$

$$-\frac{2}{14\cdot 13\cdot 12\cdot 11}\,\frac{d^4(\mu^{14})}{d\mu^4}\Big]$$

$$=\Big[\frac{1\cdot 2\ldots 12}{12\cdot 11\cdot 9\cdot\ (3\cdot 5\ \ 25)}\Big\{25\ P^4{}_{12}+21\,.\,\frac{25}{2}\,.\ P^4{}_{10}+17.\,\frac{25\cdot 23}{2\cdot 4}.\ P^4{}_8$$

$$+13\,.\ \frac{25\cdot 23\cdot 21}{2\cdot 4\cdot 6}\ P^4{}_6\ +9\cdot\frac{25\cdot 23\cdot 19}{2\cdot 4\cdot 6\cdot 8}\ P^4{}_4\ \Big\}\Big]$$

$$+\ \frac{1\cdot 2\ ..16}{16\cdot 15\cdot 14\cdot 13\ (3\cdot 5\ldots 33)}\Big\{\ 33\cdot\ P^4{}_{16}+\ 29\,.\ \ \frac{33}{2}\ .\ P^4{}_{14}$$

$$+25\,.\,\frac{33\cdot 31}{2\cdot 4}P^4{}_{12}\ +\ 21\,.\,\frac{33\cdot 31\cdot 29}{2.4.6}\ P^4{}_{10}\ +17\,.\ \frac{33\cdot 31\ldots 27}{2\cdot 4\cdot 6\cdot 8}P^4{}_8$$

$$+13\ .\ \frac{33\cdot 31\cdot 25}{2\cdot 4\ldots 10}\ P^4{}_6\ +9.\ \frac{33\cdot 31\cdot 23}{2\cdot 4\ldots 12}\ \ P^4{}_4\ \Big\}$$

$$-\frac{2\cdot 1\cdot 2\ ..14}{14\cdot 13\cdot 12\cdot 11\cdot\ (3\cdot 5\ldots 29)}\ \Big\{29\,.\ \ P^4{}_{14}$$

$$+\ldots\ldots+9\,.\ \frac{29\cdot 27\cdot 21}{2\cdot 4\cdot 6\ldots 10}\ P^4{}_4\ \ \Big\}\,.$$

III. $(1-\mu^2)^{\frac{6}{2}}\ [\mu^8\ (1-\mu^2)]\ =(1-\mu^2)^{\frac{6}{2}}\ \ [\mu^8-\mu^{10}\ \ ]$

$$= \Big[\frac{1\cdot 2 \ldots 14}{14\cdot 13\cdot 7\cdot (3\cdot 5 \ldots 29)} \Big\{ 29\cdot P^6_{14} + 25\cdot \frac{29}{2} \cdot P^6_{12} + \ldots$$

$$+ 13 \cdot \frac{29\cdot 27\cdot 23}{2\cdot 4\cdot 6\cdot 8} P^6_6 \Big\}$$

$$- \frac{1\cdot 2 \ldots 16}{16\cdot 15\cdot 11\,(3\cdot 5 \cdots 33)} \Big\{ 33\, P^6_{16} + 29 \cdot \frac{33}{2} \cdot P^6_{14} + \ldots\ldots$$

$$+ 13 \cdot \frac{33\cdot 31 \ldots 25}{2\cdot 4 \ldots 10} P^6_6 \Big\} \Big] .$$

Therefore, we see that

$$\cos^8\theta . \sin^8\theta . \sin^4\phi . \cos^4\phi$$

$$= \frac{1}{2^7} \Big[\text{I.} \cos 8\phi - 4.\ \text{II.} \cos 4\phi + 6.\ \text{III} \Big] .$$

Putting $\phi = 0$ we see that

$$0 = \text{I} - 4.\ \text{II} + 6.\ \text{III}$$

In conclusion, I wish to express my best thanks to Professor Ganesh Prasad for kindly suggesting to me this problem and for his keen interest and encouragement, while the work was in progress.

Paper III

~~11~~

"On a System of Spherical Harmonics"

BY

N. G. Shabde.

Introduction.

Bromwich * has given a new solution of Laplace's equation represented by $Y_n = \frac{\partial}{\partial n}\{r^n \cdot P_n(\cos\theta)\}$, which occurs in certain potential problems he deals with. G. N. Watson in a note,† with the title "On some solutions of Laplace's equation," has given solutions of Laplace's equation represented by

$$\frac{\partial^2}{\partial n^2}(r^n P_n), \dots\dots\dots, \frac{\partial^m}{\partial n^m}(r^n P_n).$$

Expansions for $\frac{\partial}{\partial n}\{P_n(\cos\theta)\}$ in terms of Legendre's functions have been given by Bromwich and later on by H. B. C. Darling.‡ A. F. Joliffe § has put $\frac{\partial}{\partial n}\{P_n(\cos\theta)\}$ in the form

$$\frac{2}{2^n . n!}\left(\frac{d}{d\mu}\right)^n\left\{(\mu^2-1)^n \log\frac{1+\mu}{2}\right\} - P_n . \log\frac{1+\mu}{2},\ \mu \text{ being} = \cos\theta,$$

analogous to the well-known form for

$$Q_n(\mu) = \frac{1}{2^n . n!}\left(\frac{d}{d\mu}\right)^n\left\{(\mu^2-1)^n \log\frac{1+\mu}{1-\mu}\right\} - \tfrac{1}{2} P_n \log\frac{1+\mu}{1-\mu}.$$

* *Proc. London Math. Society* (2), Vol. XII (1913), p. 100.

† *Proc. London Math. Society* (2), Vol. XII (1913), p. viii.

‡ *Quarterly Journal of Mathematics*, Vol. 49 (1923), p. 289.

§ *Messenger of Mathematics*, Vol. XLIX, p. 125.

Now it is obvious to see that since $V=r^n.P_n^m(\cos\theta)\frac{\cos}{\sin}m\phi$ is a solution of $\nabla^2 V\equiv\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$, when expressed in polar co-ordinates, $\frac{\partial^{p+q}}{\partial m^p.\partial n^q}\{r^n.P_n^m(\cos\theta).\frac{\cos}{\sin}m\phi\}$, must also give solutions of Laplace's equation. The object of the present paper is to give expressions and expansions in terms of associated Legendre functions for $\frac{\partial^{p+q}}{\partial m^p.\partial n^q}\{r^n.P_n^m.\frac{\cos}{\sin}m\phi\}$, thus obtaining a system of spherical harmonics.

§ 1.

Taking $q=1$ and $p=0$, to find the value of

$$\frac{\partial^{p+q}}{\partial m^p.\partial n^q}\left\{r^n P_n^m(\cos\theta)\frac{\cos}{\sin}m\phi\right\} \text{ or of } \frac{\partial}{\partial n}\{r^n.P_n^m\frac{\cos}{\sin}m\phi\}.$$

$$\frac{\partial}{\partial n}\{r^n P_n^m.\frac{\cos}{\sin}m\phi\}=r^n\frac{\cos}{\sin}m\phi\left[\log r.P_n^m+\frac{\partial}{\partial n}P_n^m\right] \quad \ldots \text{(I)}$$

To find the value of $\frac{\partial}{\partial n}P_n^m$:—

$$* \frac{\partial P_n}{\partial n}=P_n.\log\frac{1+\mu}{2}+A_n P_n-2\left\{\frac{2n-1}{1\cdot 2n}P_{n-1}-\frac{2n-3}{2(2n-1)}P_{n-2}+\ldots\ldots\right.$$

$$\left.\ldots\ldots+\frac{3(-1)^{n-2}}{n(n-1)}P_1+\frac{(-1)^{n-1}}{n(n+1)}\right\},$$

where $A_n=2\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{2n}\right)$,

* Bromwich, *l.c.*, or Hobson, *Theories of Spherical and Ellipsoidal Harmonics*, p. 172, art. 112.

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or *

$$\frac{\partial \mathrm{P}_n}{\partial n} = -\frac{1}{2n+1}\mathrm{P}_n + \sum_{p=0}^{\infty}\frac{(-1)^p.(2n+2p+3)}{(p+1)(2n+p+2)}\mathrm{P}_{n+p+1}$$

$$-\sum_{p=0}^{n-1}\frac{(-1)^p.(2n-2p-1)}{(p+1)(2n-p)}\mathrm{P}_{n-p-1}$$

Hence

$$\frac{\partial}{\partial n}\mathrm{P}_n^m = \frac{\partial}{\partial n}\left\{(1-\mu^2)^{\frac{m}{2}}.\frac{d^m}{d\mu^m}\mathrm{P}_n\right\} = (1-\mu^2)^{\frac{m}{2}}.\frac{d^m}{d\mu^m}\left\{\frac{\partial \mathrm{P}_n}{\partial n}\right\}$$

$$=(1-\mu^2)^{\frac{m}{2}}.\frac{d^m}{d\mu^m}\left[\mathrm{P}_n\log\frac{1+\mu}{2}+\mathrm{A}_n\mathrm{P}_n-2\left\{\frac{2n-1}{1\cdot 2n}\mathrm{P}_{n-1}-\ldots\right\}\right].$$

$$=\log\frac{1+\mu}{2}.\mathrm{P}_n^m+\mathrm{A}_n\mathrm{P}_n^m-2\left\{\frac{2n-1}{1\cdot 2n}\mathrm{P}_{n-1}^m-\frac{2n-3}{2(2n-1)}\mathrm{P}_{n-2}^m+\ \ldots\right\}$$

$$+\left[m\sqrt{\frac{1-\mu}{1-\mu}}.\mathrm{P}_n^{m-1}-\frac{m(m-1)}{1\cdot 2}.\left(\frac{1-\mu}{1+\mu}\right)\mathrm{P}_n^{m-2}+\ldots\ldots\ldots\ldots\right.$$

$$+\frac{m(m-1)(m-2)2}{1\cdot 2\cdot 3}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{3}{2}}.\mathrm{P}_n^{m-3}-\ldots+(-1)^{m-1}.\mathrm{P}_n\cdot(m-1)!\times$$

$$\left.\times\left(\frac{1-\mu}{1+\mu}\right)^{\frac{m}{2}}\right]\qquad \ldots\ \ldots\ \ldots\ \ldots\quad \text{(II)}$$

or

$$\frac{\partial}{\partial n}\mathrm{P}_n^m=(1-\mu^2)^{\frac{m}{2}}\frac{d^m}{d\mu^m}\left\{\frac{\partial \mathrm{P}_n}{\partial n}\right\}$$

$$=(1-\mu^2)^{\frac{m}{2}}\frac{d^m}{d\mu^m}\left[-\frac{1}{2n+1}\mathrm{P}_n+\sum_{p=0}^{\infty}\frac{(-1)^p(2n+2p+3)}{(p+1)(2n+p+2)}\mathrm{P}_{n+p+1}\right.$$

$$\left.-\sum_{p=0}^{n-1}\frac{(-1)^p.(2n-2p-1)}{(p+1)(2n-p)}\mathrm{P}_{n-p-1}\right]$$

* Darling, *l.c.*

$$= -\frac{1}{2n+1}\, P_n^m + \sum_{p=0}^{\infty} \frac{(-1)^p.(2n+2p+3)P_{n+p+1}^m}{(p+1)(2n+p+2)}$$

$$-\sum_{p=0}^{n-1} \frac{(-1)^p.(2n-2p-1)}{(p+1)(2n-p)} P_{n-p-1}^m \quad \cdots \quad \text{(III)}$$

Again using Jolliffe's expression for $\frac{\partial}{\partial n}\left\{P_n\right\}$ we have

$$\frac{\partial}{\partial n}\, P_n^m = (1-\mu^2)^{\frac{m}{2}}.\frac{d^m}{d\mu^m}\left[\frac{\partial}{\partial n}\, P_n\right]$$

$$= (1-\mu^2)^{\frac{m}{2}}.\frac{2}{2^n.n!}\left(\frac{d}{a\mu}\right)^{n+m}\left\{(\mu^2-1)^n.\log\frac{1+\mu}{2}\right\}$$

$$-\log\left(\frac{1+\mu}{2}\right).P_n^m - \left[m\sqrt{\frac{1-\mu}{1+\mu}}\,P_n^{m-1} - \frac{m(m-1)}{1\cdot 2}\left(\frac{1-\mu}{1+\mu}\right)P_n^{m-2}\right.$$

$$\left. + \ldots\ldots + (-1)^m.P_n.(m-1)!\left(\frac{1-\mu}{1+\mu}\right)^{\frac{m}{2}}\right] \quad \cdots \quad \text{(IV)}$$

Hence using (II), (III) and (IV) in (I) we have

$$\frac{\partial}{\partial n}\left\{r^n.P_n^m.\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\right\}$$

$$= r^n.\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\left[\log r.\,P_n^m + P_n^m.\log\frac{1+\mu}{2} + A_n P_n^m - 2\left\{\frac{2n-1}{1\cdot 2n}\,P_{n-1}^m - \ldots\right\}\right.$$

$$\left. + \left\{m\sqrt{\frac{1-\mu}{1+\mu}}P_n^{m-1} - \ldots + (-1)^{m-1}(m-1)!\,P_n\left(\frac{1-\mu}{1+\mu}\right)^{\frac{m}{2}}\right\}\right. \ldots \quad (1.1)$$

or

$$= r^m\,\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\left[\log r.P_n^m + \left\{-\frac{1}{2n+1}\,P_n^m\right.\right.$$

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$$+\sum_{p=0}^{\infty}\frac{(-1)^p(2n+2p+3)}{(p+1)(2n+p+2)}\mathrm{P}^m_{n+p+1}-\sum_{p=0}^{n-1}\frac{(-1)^p(2n-2p-1)}{(p+1)(2n-p)}\cdot\mathrm{P}^m_{n-p-1}\Big\}\Big]\quad(1.2)$$

or

$$=r^n.\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\Big[\log r.\mathrm{P}^m_n+(1-\mu^2)^{\frac{m}{2}}.\frac{2}{2^n.n!}\times$$

$$\times\left(\frac{d}{d\mu}\right)^{m+n}\left\{(\mu^2-1)^n.\log\frac{1+\mu}{2}\right\}-\mathrm{P}^m_n.\log\left(\frac{1+\mu}{2}\right)-\left\{m\sqrt{\frac{1-\mu}{1+\mu}}\,\mathrm{P}^{m-1}_n\right.$$

$$\left.-\frac{m(m-1)}{1\cdot 2}.\left(\frac{1-\mu}{1+\mu}\right)\mathrm{P}^{m-2}_n+\ldots+(-1)^{m-1}.(m-1)!\,\mathrm{P}_n.\left(\frac{1-\mu}{1+\mu}\right)^{\frac{m}{2}}\right\}\Big]\quad(1.3)$$

§ 2.

To obtain expressions for $\frac{\partial^{2p}}{\partial n^{2p}}\mathrm{P}^m_n(\mu)$ *and* $\frac{\partial^{2p+1}.\mathrm{P}^m_n(\mu)}{\partial n^{2p+1}}$ *and thus to find expressions for* $\frac{\partial^{2p}}{\partial n^{2p}}\left\{r^n.\mathrm{P}^m_n.\begin{matrix}\cos\\ \sin\end{matrix}\,m\phi\right\}$ *and*

$$\frac{\partial^{2p+1}}{\partial n^{2p+1}}\left\{r^n.\mathrm{P}^m_n.\begin{matrix}\cos\\ \sin\end{matrix}\,m\phi\right\}$$

G. N. Watson in his note * has obtained values for $\theta_{2p,n}$ and $\theta_{2p+1,n}$

where $\theta_{2p,n}=\frac{1}{(2p)!}\frac{\partial^{2p}}{\partial n^{2p}}\mathrm{P}_n$ and $\theta_{2p+1,n}=\frac{1}{(2p+1)!}.\frac{\partial^{2p+1}}{\partial n^{2p+1}}\mathrm{P}_n$.

According to him,

$$\theta_{2p,n}=\sum_{k=0}^{2p}v_k.\mathrm{T}_{2p-k}\text{ and }\theta_{2p+1,n}=\sum_{k=0}^{2p+1}u_k.\mathrm{T}_{2p+1-k}$$

where

$$u_p=\sum_{q=1}^{n}\delta_{p,q}(-1)^{(p-1)q}\mathrm{P}_{n-q}(\mu),\; v_p=\sum_{q=1}^{n}\delta_{p,q}\,(-1)^{pq}\mathrm{P}_{n-q}(\mu),$$

$\delta_{p,q}$ depending on n, p and q only (not on μ) and

$$\mathrm{T}_0=1,\;\mathrm{T}_1=\log(1+\tfrac{1}{2}t),\;\mathrm{T}_2=\sum_{s=1}^{\infty}\frac{(-1)^{s-1}.t^s}{2^s.s^2},$$

* *L.c.*

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$$T_{2p+1}=\sum_{s=p+1}^{\infty}\frac{(-1)^{s+p+1}}{2^s.s.}\sigma_{s-1,p}.\text{ and }T_{2p+2}=\sum_{s=p+1}^{\infty}\frac{(-1)^{s+p+1}}{2^s.s^2}\sigma_{s-1,p}.t^s$$

where $t=\mu-1=-2\sin^2\frac{\theta}{2}$, while $\sigma_{s,p}$ is the sum of the products p at a time of 1^{-2}, 2^{-2}, ... s^{-2}, the functions T_p having no singularities in the real space except on the negative part of the axis of harmonics, also $t.T'_{2p+2}=T_{2p+1}$, $(t+2)$. $T'_{2p+1}=T_{2p}$. Expressions for $\delta_{p,q}$ in terms of n, p and q have been obtained in Watson's note.

So we see that as

$$\frac{\partial^{2p}P_n}{\partial n^{2p}}=(2p)!\sum_{k=0}^{2p}.v_k.T_{2p-k}$$

$$\frac{\partial^{2p}P_n^m}{\partial n^{2p}}=(1-\mu^2)^{\frac{m}{2}}.\frac{d^m}{d\mu^m}\left\{\frac{\partial^{2p}}{\partial n^{2p}}P_n\right\}$$

$$=(2p)!\,(1-\mu^2)^{\frac{m}{2}}.\frac{d^m}{d\mu^m}\left[\sum_{k=0}^{2p}v_k.T_{2p-k}\right]$$

$$=(2p)!\,(1-\mu^2)^{\frac{m}{2}}.\frac{d^m}{d\mu^m}\left[\sum_{k=0}^{2p}\left\{T_{2p-k}\left(\sum_{q=1}^{n}\delta_{k,q}(-1)^{kq}P_{n-q}(\mu)\right)\right\}\right]$$

$$=(2p)!\sum_{k=0}^{2p}\Big[T_{2p-k}\sum_{q=1}^{n-m}(-1)^{kq}.\delta_{k,q}.P^m_{n-q}(\mu)$$

$$+{}^mC_1.T'_{2p-k}.\sqrt{1-\mu^2}\sum_{q=1}^{n-m+1}(-1)^{kq}.\delta_{k,q}.P^{m-1}_{n-q}(\mu)$$

$$+{}^mC_2.T''_{2p-k}.(1-\mu)^2.\sum_{q=1}^{n-m+2}(-1)^{kq}.\delta_{k,q}.P^{m-2}_{n-q}(\mu)$$

$$+\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad+$$

$$+T^m_{2p-k}(1-\mu^2)^{\frac{m}{2}}.\sum_{q=1}^{n}(-1)^{kq}.\delta_{k,q}P_{n-q}(\mu)\Big]\qquad\ldots\quad(I)$$

Similarly, since ~~therefore~~

$$\frac{\partial^{2p+1}}{\partial n^{2p+1}} \mathrm{P}_n(\mu)=(2p+1)!\sum_{k=0}^{2p+1} u_k\cdot \mathrm{T}_{2p+1-k}$$

$$\frac{\partial^{2p+1}}{\partial n^{2p+1}} \mathrm{P}_n^m(\mu)=(2p+1)!\,(1-\mu^2)^{\frac{m}{2}}\frac{d^m}{d\mu^m}\left[\sum_{k=0}^{2p+1} u_k\cdot \mathrm{T}_{2p+1-p}\right].$$

$$=(2p+1)!\,(1-\mu^2)^{\frac{m}{2}}\frac{d^m}{d\mu^m}\left[\sum_{k=0}^{2p+1}\left\{\mathrm{T}_{2p+1-k}\left(\sum_{q=1}^{n}\delta_{k,q}(-1)^{(k-1)q}\,\mathrm{P}_{n-q}(\mu)\right)\right\}\right]$$

$$=(2p+1)!\sum_{k=0}^{2p+1}\left[\mathrm{T}_{2p+1-k}\sum_{q=1}^{n-m}\delta_{k,q}(-1)^{(k-1)q}.\mathrm{P}_{n-q}^m(\mu)\right.$$

$$+{}^m\mathrm{C}_1.\mathrm{T}_{2p-k+1}\sqrt{1-\mu^2}\,.\sum_{q=1}^{n-m+1}(-1)^{(k-1)q}.\mathrm{P}_{n-q}^{m-1}(\mu)$$

$$+{}^m\mathrm{C}_2\cdot\mathrm{T}''_{2p-k+1}(1-\mu^2)\sum_{q=1}^{n-m+2}(-1)^{(k-1)q}\delta_{k,q}\mathrm{P}_{n-q}^{m-2}(\mu)$$

$$+\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad+$$

$$\left.+\mathrm{T}^m_{2p+1-k}.(1-\mu^2)^{\frac{m}{2}}.\sum_{q=1}^{n}(-1)^{(k-1)q}.\delta_{k,q}.\mathrm{P}_{n-q}(\mu)\right] \qquad \ldots \text{(II)}$$

Therefore

$$\frac{\partial^c}{\partial n^c}\left\{r^n.\ \mathrm{P}_n^m(\mu)\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\right\}$$ where c is an even integer $2p$ or an odd integer $2p+1$

$$=\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\,(1-\mu^2)^{\frac{m}{2}}.\frac{d^m}{d\mu^m}\left\{\frac{\partial^c}{\partial n^c}\left(r^n.\mathrm{P}_n(\mu)\right)\right\}$$

$$=\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi.(1-\mu^2)^{\frac{m}{2}}.\frac{d^m}{d\mu^m}\left\{\sum_{l=0}^{c}\frac{r^n(\log r)^{c-l}c!}{(c-l)!\,l!}\,\frac{\partial^l}{\partial n^l}\mathrm{P}_n(\mu)\right\}$$

$$= \frac{\cos}{\sin} m\phi . \sum_{l=0}^{c} \frac{r^n . (\log r)^{c-l} c\,!}{(c-l)\,!\; l\,!} . \left\{ (1-\mu^2)^{\frac{m}{2}} . \frac{d^m}{d\mu^m} \left(\frac{\partial^l}{\partial n^l} P_n \right) \right\} . \quad \ldots \ (2{\cdot}1)$$

This can be further evaluated in terms of associated Legendre's functions by using (I) and (II).

§ 3.

To derive an expression for $\frac{\partial}{\partial m} P_n^m(\mu)$ *in terms of Legendre's functions from an expression given by H. B. C. Darling in a different notation.*

As in Darling's paper* let $P_{m,s}(\mu)$ denote the co-efficient of a^m in the expansion of $(1-2\mu a+a^2)^{\frac{1}{2}(2s-1)}$. Then according to him,

$$\frac{dP_{m,s}}{ds} = A_{m,s} . P_{m,s} - \sum_{p=1}^{p} \frac{2m-2s-4p+1}{p(2m-2s-2p+1)} . P_{m-2p,s} . \quad \ldots \ (\mathrm{I})$$

$$\text{where } A_{m,s} = 2\left(\frac{1}{2m-2s+1} + \frac{1}{2m-2s+3} + \ldots + \frac{1}{2s-1} \right) \text{ if } m < 2s$$

$$\text{and} \quad = -2\left(\frac{1}{2m-2s-1} + \frac{1}{2m-2s-3} + \ldots\ldots + \frac{1}{2s+1} \right) \text{ if } m > 2s,$$

the series terminating when $m-2p$ is unity or zero.

Now the co-efficient of h^{n-m} in the expansion of

$$\frac{1}{(1-2\mu h+h^2)^{m+\frac{1}{2}}} \text{ in powers of } h \text{ is } \frac{2^m . m\,!\,(1-\mu^2)^{-\frac{1}{2}m} . P_n^m(\mu)}{(2m)\,!},$$

$P_n^m(\mu)$ being Legendre's associated function. So putting $s=-s$, $m=n-s$ and $a=h$ in $(1-2\mu a+a^2)^{\frac{1}{2}(2s-1)}$ we see that

$$P_{n-s,-s}(\mu) = \frac{2^s . s\,!\,(1-\mu^2)^{-\frac{1}{2}s} . P_n^s(\mu)}{(2s)\,!}$$

* L.c.

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or

$$\mathrm{P}_{n-m,-m}(\mu)=\frac{2^m \cdot m\,!\,(1-\mu^2)^{-\frac{1}{2}m}\,\mathrm{P}_n^m(\mu)}{(2m)\,!}$$

Moreover,

$$\mathrm{P}_{m+2s,s}=\frac{2^{-2s}(m\,!)\{(2s)\,!\}^2\cdot(1-\mu^2)^s\cdot\mathrm{P}_{m,-s}}{(m+2s)!(s\,!)^2}$$

$$\mathrm{P}_{n+s,s}=\frac{2^{-2s}.(n-s)\,!\{(2s)!\}^2.(1-\mu^2)^s.\mathrm{P}_{n-s,-s}}{(n+s)!(s!)^2}$$

$$=\frac{2^{-2s}(n-s)!\{(2s)!\}^2.(1-\mu^2)^s.2^s.s!(1-\mu^2)^{-\frac{1}{2}s}\ \mathrm{P}_n^s(\mu)}{(n+s)!(s!)^2.\quad(2s)!}$$

$$=\frac{(1-\mu^2)^{\frac{1}{2}s}.(2s)!(n-s)!\,\mathrm{P}_n^s(\mu)}{(n+s)!\ s!\ 2^s}$$

$$\mathrm{P}_{n+m,m}=\frac{(1-\mu^2)^{\frac{1}{2}m}\ (2m)!(n-m)!}{(n+m)!\ m!\ 2^m}\ \mathrm{P}_n^m(\mu) \qquad \dots \quad \text{(II)}$$

(I) can be written as

$$\frac{d\mathrm{P}_{m+n,n}}{dm}=\mathrm{A}_{m+n,m}\mathrm{P}_{m+n,m}-\sum_{p=1}^{p}\frac{2n-4p+1}{p(2n-2p+1)}\times\mathrm{P}_{n+m-2p,n}$$

Substituting in this from (II) we have

$$\frac{d}{dm}\left[\frac{(1-\mu^2)^{\frac{m}{2}}.(2m)!(n-m)!\mathrm{P}_n^m(\mu)}{(n+m)!m!2^m}\right]$$

$$=\mathrm{A}_{m+n,m}\left[\frac{(1-\mu^2)^{\frac{m}{2}}.(2m)!(n-m)!}{(n+m)!m!\,2^m}\ \mathrm{P}_n^m(\mu)\right]$$

$$-\sum_{p=1}^{p}\frac{2n-4p+1}{p(2n-2p+1)}\left[\frac{(1-\mu^2)^{\frac{1}{2}m}(2m)!(n-2p-m)!}{(n-2p+m)!\ m!\ 2^m}\mathrm{P}_{n-2p}^m(\mu)\right].$$

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Hence,

$$\frac{\partial}{\partial m} \mathrm{P}_n^m(\mu) = \frac{m!(n+m)!\,2^m}{(1-\mu^2)^{\frac{1}{2}m}(2m)!\,(n-m)!} \times$$

$$\times \Bigg[\frac{(1-\mu^2)^{\frac{1}{2}m}.\,(2m)!\,(n-m)!}{(n+m)!\,m!\,2^m} \times \mathrm{A}_{m+n,m} \times \mathrm{P}_n^m(\mu) - \mathrm{P}_n^m(\mu) \times$$

$$\frac{\partial}{\partial m} \left\{ \frac{(1-\mu^2)^{\frac{1}{2}m}(2m)!(n-m)!}{m!(n+m)!\,2^m} \right\} - \sum_{p=1}^{p} \frac{2n-4p+1}{p(2n-2p+1)} \times$$

$$\times \left\{ \frac{(1-\mu^2)^{\frac{1}{2}m}.(2m)!(n-2p-m)!}{m!(n-2p+m)!\,2^m} \mathrm{P}_{n-2p}^m(\mu) \right\} \Bigg]$$

$$= \mathrm{A}_{m+n,m} \mathrm{P}_n^m(\mu) - \sum_{p=1}^{p} \frac{2n-4p+1}{p(2n-2p+1)} \; \frac{(n+m)!(n-2p-m)!}{(u-m)!(n+m-2p)!} \times P_{n-2p}^m(\mu)$$

$$\cancel{\times \mathrm{P}_{n-2p}^m(\mu)} - \tfrac{1}{2} \log (1-\mu^2).\mathrm{P}_n^m(\mu)$$

$$- \mathrm{P}_n^m(\mu). \frac{\frac{\partial}{\partial m} \left\{ \frac{(2m)!(n-m)!}{m!(n+m)!2^m} \right\}}{\frac{2m!(n-m)!}{m!(n+m)!\,2^m}}$$

$$= \mathrm{P}_n^m(\mu) \left[\mathrm{A}_{m+n,m} - \tfrac{1}{2} \log (1-\mu^2) - \{ \psi(2m+1) - \psi(n-m+1) \right.$$

$$\left. - \psi(n+m+1) - \psi(m+1) - \log 2 \} \right]$$

$$- \sum_{p=1}^{p} \frac{(2n-4p+1)}{p(2n-2p+1)} \; \frac{(n+m)!\; n-2p-m)!}{(n-m)!(n+m-2p\,!} \mathrm{P}_{n-2p}^m \quad \ldots \quad (3.1)$$

$$\frac{\partial}{\partial m}. \left\{ r^n.\, \mathrm{P}_n^m(\mu) \; \begin{matrix}\cos\\ \sin\end{matrix} \, m\phi \right\}$$

$$= r^n \; \begin{matrix}\cos\\ \sin\end{matrix} \, m\phi \,.\, \frac{\partial}{\partial m} \mathrm{P}_n^m + \phi.\, r^n.\, \mathrm{P}_n^m. \left\{ \begin{matrix} -\sin \\ +\cos \end{matrix} \, m\phi \right\}$$

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$$=r^n.\ \begin{matrix}\cos\\ \sin\end{matrix}\ m\phi.\ \Big[\mathrm{P}_n^m(\mu)\{\Delta_{m+n,m}-\tfrac{1}{2}\log(1-\mu^2)-\Big(\psi(2m+1)$$

$$+\psi(n-m+1)-\psi(m+1)-\psi(n+m+1)-\log 2\Big)\Big\}$$

$$-\sum_{p=1}^{p}\frac{2n-4p+1}{p(2n-2p+1)}\ \frac{(n+m)!(n-2p-m)!}{(n-m)!(n+m-2p)!}.\ \mathrm{P}_{n-2p}^m\Big]$$

$$+\phi.\ r^n.\ \mathrm{P}_n^m(\mu).\left\{\begin{matrix}-\sin\\ +\cos\end{matrix}\ m\phi\right\}\quad\ldots\quad(3.2)$$

§ 4.

To give another expression for $\dfrac{\partial}{\partial m}\,\mathrm{P}_n^m(\mu)$ *and expressions for*

$$\frac{\partial^2}{\partial m^2}\,\mathrm{P}_n^m,\ \frac{\partial^3}{\partial m^3}\mathrm{P}_n^m,\ldots;\ \frac{\partial^{2p}}{\partial m^{2p}}\mathrm{P}_n^m,\ \frac{\partial^{2p+1}}{\partial m^{2p+1}}\,\mathrm{P}_n^m,$$

in terms of associated Legendre functions and thus to evaluate

$$\frac{\partial^c}{\partial m^c}\ \left\{r^n.\ \mathrm{P}_n^m.\ \begin{matrix}\cos\\ \sin\end{matrix}\ m\phi\right\}\ \textit{where } c=2p \text{ or } 2p+1.$$

If $\mu>1$ we have for m an integer

$$\mathrm{P}_n^m(\mu)=\frac{\Pi(n+m)}{\Pi(n)}\cdot\frac{1}{\pi}\cdot\int_0^{\pi}\{\mu+\sqrt{\mu^2-1}.\ \cos\phi\}^n\cos\ m\phi d\phi$$

and if $\mu<1$

$$\mathrm{P}_n^m(\mu)=\frac{\Pi(n+m)}{\Pi(n)}\cdot\frac{e^{-\frac{m\pi}{2}i}}{\pi}\int_0^{\pi}\{\mu+\sqrt{u^2-1}\ \cos\phi\}^n\cos m\phi d\phi.$$

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$$\frac{\partial}{\partial m} P_n^m(\mu)$$

$$=\frac{\frac{\partial}{\partial m}\{\Pi(n+m)\}}{\Pi(n)}\cdot\frac{1}{\pi}\int_0^{\pi}\{\mu+\sqrt{\mu^2-1}\cos\phi\}^n\cos m\phi d\phi$$

$$-\frac{\Pi(n+m)}{\Pi(n)}\cdot\frac{1}{\pi}\int_0^{\pi}\{\mu+\sqrt{\mu^2-1}\cos\phi\}^n\sin m\phi.\,\phi\,d\phi \text{ if } \mu>1$$

and

$$=\frac{\frac{\partial}{\partial m}\left\{\Pi(m+n)e^{-\frac{m\pi}{2}i}\right\}}{\Pi(n)}\cdot\frac{1}{\pi}\int_0^{\pi}\{\mu+\sqrt{\mu^2-1}\cos\phi\}^n\cos m\phi.\,d\phi$$

$$-\frac{\Pi(n+m)e^{-\frac{m\pi}{2}i}}{\Pi(n)}\cdot\frac{1}{\pi}\int_0^{\pi}\{\mu+\sqrt{\mu^2-1}\cos\phi\}^n\sin m\phi.\,\phi d\phi$$

if $\mu<1$.

Now

$$\phi=2\{\sin\phi-\tfrac{1}{2}\sin 2\phi+\tfrac{1}{3}\sin 3\phi-\quad\ldots\},$$

$$-\pi+\delta\leq\phi\leq\pi-\delta,\ \delta>0.$$

Multiplying both the sides by $\sin m\phi$,

$$\phi\sin m\phi=\{\cos(m-1)\phi-\cos(m+1)\phi\}-\tfrac{1}{2}\{\cos(m-2)\phi$$

$$-\cos(m+2)\phi\}+\ldots\ldots,-\pi\leq\phi\leq\pi$$

Therefore

$$\frac{\partial}{\partial m} P_n^m(\mu)$$

$$=\psi(n+m+1)P_n^m(\mu)-\Pi(n+m)\left[\left\{\frac{P_n^{m-1}(\mu)}{\Pi(n+m-1)}-\frac{P_n^{m+1}(\mu)}{\Pi(n+m+1)}\right\}\right.$$

$$-\tfrac{1}{2}\left\{\frac{P_n^{m-2}(\mu)}{\Pi(n+m-2)}-\frac{P_n^{m+2}(\mu)}{\Pi(n+m+2)}\right\}$$

$$-\tfrac{1}{3}\left\{\frac{P_n^{m-3}(\mu)}{\Pi(n+m-3)}-\frac{P_n^{m+3}(\mu)}{\Pi(n+m+3)}\right\}-\ldots\Bigg]$$

if $\mu>1$

and

$$=\{\psi(n+m+1)-\frac{\pi}{2}i\}P_n^m(\mu)$$

$$-\Pi(n+m)\Bigg[\left\{\frac{e^{\frac{\pi}{2}i}P_n^{m-1}(\mu)}{\Pi(n+m-1)}-\frac{e^{-\frac{\pi}{2}i}P_n^{m+1}(\mu)}{\Pi(n+m+1)}\right.$$

$$-\frac{1}{2}\left\{\frac{P_n^{m-2}(\mu).\ e^{\pi i}}{\Pi(n+m-2)}-\frac{P_n^{m-2}.e^{-\pi i}}{\Pi(n+m+2)}\right\}$$

$$+\tfrac{1}{3}\left\{\frac{e^{-\frac{3\pi}{2}i}P_n^{m-3}(\mu)}{\Pi(n+m-3)}-\frac{e^{\frac{3\pi}{2}i}P_n^{m+2}(\mu)}{\Pi(n+m+3)}\right\}-\ldots\Bigg]\text{ if }\mu<1 \quad (4.1)$$

So we see that

$$\frac{\partial}{\partial m}\{r^n.\ P_n^m.\ {\cos \atop \sin} m\phi\}$$

$$=r^n.\ P_n^m(\mu).\ \phi\{{-\sin \atop +\cos} m\phi\}+r^n\frac{\partial}{\partial m}\{P_n^m(\mu)\}.{\cos \atop \sin} m\phi \quad \ldots \quad (4.2)$$

and can be evaluated by using the expansion for $\frac{\partial}{\partial m}P_n^m(\mu)$ given in (4.1)

Again*

$$\frac{1}{4}\phi^2=\frac{1}{12}\pi^2-\sum_{p=1}^{\infty}\frac{(-1)^{p-1}.\cos p\phi}{p^2}, \qquad (-\pi\leq\phi\leq\pi)$$

* *Modern Analysis*, p. 163, Ex. 2.

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$$\phi^2 \cos m\phi = \tfrac{1}{3}\pi^2 . \cos m\phi - 2 \sum_{p=1}^{\infty} \frac{(-1)^{p-1} . \{\cos (m+p)\phi + \cos (m-p)\phi\}}{p^2}$$

$$\frac{\partial^2 P_n^m(\mu)}{\partial m^2} = \frac{\partial^2}{\partial m^2} \left[\frac{\Pi(n+m)}{\Pi(n)} . \frac{1}{\pi} . \int_0^\pi \{\mu + \sqrt{\mu^2-1} . \cos\phi\}^n \cos m\phi . \, d\phi \right]$$

$$= \frac{\partial^2}{\partial m^2} \left\{ \frac{\Pi(n+m)}{\Pi(n)} \right\} . \frac{1}{\pi} . \int_0^\pi \{\mu + \sqrt{\mu^2-1} . \cos\phi\}^n . \cos m\phi . \, d\phi$$

$$-2 \, \frac{\frac{\partial}{\partial m}\{\Pi(n+m)\}}{\Pi(n)} . \frac{1}{\pi} . \int_0^\pi \{\mu + \sqrt{\mu^2-1} . \cos\phi\}^n . \phi . \sin m\phi \, d\phi$$

$$- \frac{\Pi(n+m)}{\Pi(n)} . \frac{1}{\pi} \int_0^\pi \{\mu + \sqrt{\mu^2-1} \cos\phi\}^n . \cos m\phi . \phi^2 d\phi$$

$$= \frac{\frac{\partial^2}{\partial m^2}\{\Pi(n+m)\}}{\Pi(n+m)} . P_n^m - 2 \frac{\partial}{\partial m} \{\Pi(n+m)\}$$

$$\times \left[\left\{ \frac{P_n^{m-1}}{\Pi(n+m-1)} - \frac{P_n^{m+1}}{\Pi(n+m+1)} \right\} - \frac{1}{2} \left\{ \qquad \right\} + \dots \right]$$

$$+ \Pi(n+m) \left[\tfrac{1}{3}\pi^2 . \frac{P_n^m}{\Pi(n+m)} - 2 \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p^2} \right.$$

$$\left. \times \left\{ \frac{P_n^{m+p}}{\Pi(n+m+p)} + \frac{P_n^{m-p}}{\Pi(n+m-p)} \right\} \right] \text{ if } \mu > 1$$

and

$$= \frac{\frac{\partial^2}{\partial m^2} \left\{ \Pi(n+m) e^{\frac{m\pi}{2} i} \right\}}{\Pi(n+m)} . e^{\frac{m\pi}{2} i} . P_n^m$$

$$-2 \, \frac{\partial}{\partial m} \left\{ \Pi(n+m) e^{-\frac{m\pi}{2} i} \right\} \left[\left\{ \frac{e^{-(m+1)\frac{\pi}{2} i} \, P^{m-1}}{\Pi(n+m-1)^n} \right. \right.$$

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$$+\frac{P_n^m . e^{-(m+1)\frac{\pi}{2}i}}{\Pi(n+m+1)}\Big\}\Big]-\frac{1}{2}\Big\{\qquad\Big\}+\ldots$$

$$+\Pi(n+m)e^{-\frac{\pi m}{2}i}\left[\frac{\frac{1}{3}\pi^2 . P_n^m . e^{\frac{m\pi}{2}i}}{\Pi(n+m)}-2\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{p^2}\right.$$

$$\times\left\{\frac{e^{(m+p)\frac{\pi}{2}} . P_n^{m+p}}{\Pi(n+m+p)}+\frac{e^{(m-p)\frac{\pi}{2}i} . P_n^{m-p}}{\Pi(m+n-p)}\right\}\text{if } \mu<1 \quad \ldots \quad (4.3)$$

Hence

$$\frac{\partial^2}{\partial m^2}\left\{r^n . P_n^m . \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\right\}$$

$$=r^n\left\{\frac{\partial^2}{\partial m^2}\, P_n^m . \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi+2\phi . \left\{\begin{matrix}-\sin\\ +\cos\end{matrix}\, m\phi\right\} . \frac{\partial}{\partial m}P_n^m\right.$$

$$\left.+\,\phi^2\left\{\begin{matrix}-\cos\\ +\sin\end{matrix}\, m\phi\right\} . P_n^m\right\} \quad \ldots \quad (4.4)$$

and can therefore be evaluated by using (4.1) and (4.3).

It can be easily shown that

$$\phi^3\sin m\phi=6\sum_{n=1}^{\infty}(-1)^{n-1} . \left(\frac{\pi^2}{6}-\frac{1}{n^2}\right)\frac{\cos(m-n)\phi-\cos(m+n)\phi}{n};-\pi\leq\phi\leq\pi.$$

$$\frac{\partial^3}{\partial m^3}\, P_n^m$$

$$=\frac{\frac{\partial^3}{\partial m^3} . \left\{(\Pi(n+m)\left(1 \text{ or } e^{-\frac{m\pi}{2}i}\right)\right\}}{\Pi(n+m)} . P_n^m(\mu)\left\{1 \text{ or } e^{\frac{m\pi}{2}i}\right\}$$

$$-3 . \frac{\partial^2}{\partial m^2}\left\{\left(\Pi(n+m)(1\text{or } e^{-\frac{m\pi}{2}}\right)\right\}\times\left[\left\{\frac{P_n^{m-1}(\mu) . (1\text{or } e^{-(m-1)\frac{\pi}{2}i})}{\Pi(n+m-1)}\right.\right.$$

$$-\frac{P_n^{m+1}(\mu)\left(1\text{or } e^{-(m+-)\frac{\pi}{2}i}\right)}{\Pi(n+m+1)}\Bigg\}-\tfrac{1}{2}\{\quad\}+\tfrac{1}{3}\{\quad\}+\cdots\Bigg]$$

$$-3.\frac{\partial}{\partial m}\left\{\Pi(n+m)\left(1\text{ or } e^{\frac{m\pi}{2}i}\right)\right\}\left[\tfrac{1}{3}\pi^2.\frac{P_n^m.\left(1\text{ or } e^{-\frac{m\pi}{2}i}\right)}{\Pi(n+m)}\right.$$

$$-2\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{p^2}\left\{\frac{P_n^{m+p}\left(1\text{ or } e^{\frac{m+p}{2}\pi i}\right)}{\Pi(n+m+p)}\frac{\left(1\text{ or } e^{\frac{m-p}{2}\pi i}.P_n^{m-p}\right.}{\Pi(n+m-y}\right\}\Bigg]$$

$$+\Pi(n+m)\left(1\text{ or } e^{\frac{-m\pi}{2}i}\right)6\sum_{p=1}^{\infty}\left[\frac{(-1)^{p-1}}{p}\left(\frac{\pi^2}{6}-\frac{1}{p^2}\right)\right\{\frac{P_n^{m-p}(1\text{ or } e^{\frac{m-p}{2}\pi i})}{\Pi\ n+m+p)}$$

$$-\frac{P_n^{m+p}.(1\text{ or } e^{\frac{m+p}{2}\pi i})}{\Pi(n+m+p)}\Bigg\}\Bigg]$$

according as $\mu>1$ or <1 ... (4.5)

Hence

$$\frac{\partial^3}{\partial m^3}\ \{r^n.P_n^m.\ {\cos \atop \sin} m\phi\}$$

$$=r^n\left[{\cos \atop \sin} m\phi.\frac{\partial^3}{\partial m^3}\ P_n^m+3\left\{{-\sin \atop +\cos} m\phi\right\}.\phi.\ \frac{\partial^2}{\partial m^2}\ P_n^m\right.$$

$$\left.-3\phi^2\left\{{-\cos \atop +\sin} m\phi\right\}.\ \frac{\partial}{\partial m}P_n^m-\phi^3\left\{{-\sin \atop +\cos} m\phi.\ P_n^m\right.\right] \quad (4.6)$$

and can be evaluated by using (4.1), (4.3) and (4.5). We can easily work out that

$$\phi^{2p}.\cos m\phi=\frac{\pi^{2p}}{2p+1}.\cos m\phi+\frac{1}{\pi}.\sum_{q=1}^{\infty}(-1)^q.\left[\frac{2p.\pi^{2p-1}}{q^2}\right.$$

$$\left.-\frac{(2p-1)(2p-2)}{q^4}\ \pi^{2p-3}+\ldots-\frac{(2p)!\ \pi}{q^{2p}}\right]\{\cos\ (m+q)\phi+\cos\ (m-q)\phi\}$$

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and

$$\phi^{2p+1}.\sin m\phi = \frac{1}{\pi}\sum_{q=1}^{\infty}(-1)^q.\Big[\frac{-\pi^{2p+1}}{q} + \frac{(2p+1)(2p)\pi^{2p-1}}{q^3}$$

$$- \ \ldots \ + \frac{(2p+1)\,!\,\pi}{q^{2p+1}}\Big](\cos(m-q)\phi - \cos(m+q)\phi)$$

By using these expansions it is easy to evaluate

$\dfrac{\partial^c}{\partial m^c}.\ P^m_n$, c being $2p$ or $2p+1$ as

$$\frac{\partial^c}{\partial m^c}\ P^m_n = \frac{\partial^c}{\partial m^c}\left\{\Pi(n+m)(1 \text{ or } e^{\frac{-m\pi}{2}i})\right\}$$

$$\times\frac{1}{\Pi(n).\pi}\int_0^{\pi}\{\mu+\sqrt{\mu^2-1}\ \cos\phi\}^n \cos m\phi\, d\phi$$

$$+c.\frac{\partial^{c-1}}{\partial m^{c-1}}\left\{\Pi(n+m)(1 \text{ or } e^{\frac{-m\pi}{2}i})\right\}\frac{1}{\Pi(n)}.\frac{1}{\pi}$$

$$\times\int_0^{\pi}\{\mu+\sqrt{\mu^2-1}\ \cos\phi\}^n\{-\phi\sin m\phi\}d\phi$$

$$+\frac{c(c-1)}{1.2}\ \frac{\partial^{c-2}}{\partial m^{c-2}}\left\{\Pi(n+m)(1 \text{ or } e^{\frac{-m\pi}{2}i})\right\}.\frac{1}{\pi.\ \Pi(n)}$$

$$\times\int_0^{\pi}\{\mu+\sqrt{\mu^2-1}\ \cos\phi\}^n(-\phi^2\cos m\phi\}d\phi+\ldots$$

$$+\frac{\Pi(n+m)}{\Pi(n)}.\frac{\left(1 \text{ or } e^{\frac{-m\pi}{2}i}\right)}{\pi}\int_0^{\pi}\{\mu+\sqrt{\mu^2-1}\ \cos\phi\}^n$$

$$\left\{\frac{\partial^c}{\partial m^c}(\cos m\phi\right\}d\phi$$

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$$=\frac{\frac{\partial^{c}}{\partial m^{c}}\left\{(\Pi n+m)\left(1 \text{ or } e^{\frac{-m\pi}{2}i}\right)\right\}}{\Pi(n+m)} P_n^m(\mu)\left(1 \text{ or } e^{\frac{m\pi i}{2}}\right)$$

$$-c.\frac{\partial^{c-1}}{\partial m^{c-1}}\left\{\Pi(n+m)\left(1 \text{ or } e^{-\frac{m\pi}{2}i}\right)\right\}\left[\left(\frac{e^{-m\pi\frac{\pi}{2}i}P_n^{m-1}}{\Pi(n+m-1)}\right.\right.$$

$$\left.\left.-\frac{e^{(m+1)\frac{\pi}{2}i}P_n^{m+1}}{\Pi(n+m+1)}\right)-\tfrac{1}{2}(\quad)+\tfrac{1}{3}(\quad)+\ldots\right]$$

$$-\frac{c(c-1)}{1.2}.\frac{\partial^{c-2}}{\partial m^{c-2}}\left\{\Pi(n+m)\left(1 \text{ or } e^{\frac{-m\pi}{2}i}\right)\right\}\left[\frac{\frac{1}{3}\pi^2.e^{\frac{m\pi}{2}}.P_n^m}{\Pi(n+m)}\right.$$

$$\left.-\sum_{q=1}^{\infty}\frac{(-1)^{q-1}}{q^2}\left\{\frac{P^{m+q}\ e^{(m+q)\frac{\pi}{2}i}}{\Pi(m+n+q)}+\frac{P_n^{m-q}.e^{(m-q)\frac{\pi}{2}i}}{\Pi(n+m-q)}\right\}\right]+\ldots$$

$$\text{according as } \mu > \text{or} < 1 \qquad \ldots \quad (4.7)$$

Hence

$$\frac{\partial^{c}}{\partial m^{c}}\left\{r^n.P_n^m.\begin{matrix}\cos\\ \sin\end{matrix}\ m\phi\right\}$$

$$=r^n\left[\begin{matrix}\cos\\ \sin\end{matrix}\ m\phi.\ \frac{\partial^{c}}{\partial m^{c}}P_n^m+c.\phi.\left\{\begin{matrix}-\sin\\ +\cos\end{matrix}\ m\phi\right\}.\frac{\partial^{c-1}}{\partial m^{c-1}}P_n^m+\ldots\right.$$

$$\left.+\frac{\partial^{c}}{\partial m^{c}}\left(\begin{matrix}\cos\\ \sin\end{matrix}\ m\phi\right).P_n^m.\right] \qquad \ldots \quad (4.8)$$

and can be expressed in terms of associated Legendre functions by using (4.1), (4.3), (4.5) and (4.7).

§ 5.

To give in general the value of

$$\frac{\partial^{p+q}}{\partial n^{p+q}}.\left\{r^n.P_n^m.\begin{matrix}\cos\\ \sin\end{matrix}\ m\phi\right\}$$

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in terms of associated Legendre functions seems to become very complicated. Even taking the simplest case of

$$\frac{\partial^2}{\partial m \partial n} \left\{ r^n . P_n^m . \begin{matrix}\cos\\ \sin\end{matrix} m\phi \right\},$$

we see that the expression for this will involve double series. Thus

$$\frac{\partial^2}{\partial n \partial m} \left\{ r^n . P_n^m . \begin{matrix}\cos\\ \sin\end{matrix} m\phi \right\}$$

$$= \frac{\partial}{\partial m} \left[r^n . \begin{matrix}\cos\\ \sin\end{matrix} m\phi . \left\{ \log r . P_n^m + \frac{\partial}{\partial n} P_n^m \right\} \right]$$

$$= r^n \phi \left\{ \begin{matrix}-\sin\\ \cos\end{matrix} m\phi \right\} . \left\{ \log r . P_n^m \right\} + r^n . \begin{matrix}\cos\\ \sin\end{matrix} m\phi$$

$$\times \left\{ \log r . \frac{\partial}{\partial m} P_n^m + \frac{\partial^2}{\partial n \partial m} P_n^m \right\} \quad \dots \quad (5.1)$$

For $\frac{\partial}{\partial m} \{P_n^m\}$ we shall substitute its value from (3.1) or (4.1).

To find $\frac{\partial^2}{\partial m \partial n} \{P_n^m\}$ we have

$$\frac{\partial^2}{\partial m \partial n} P_n^m = \frac{\partial}{\partial m} \left\{ \frac{\partial}{\partial n} P_n^m \right\}$$

$$= \frac{\partial}{\partial m} \left[-\frac{1}{2n+1} P_n^m + \sum_{p=0}^{\infty} \frac{2n+2p+3}{(p+1)(2n+p+2)} . P_{n+p+1}^m \right.$$

$$\left. + \sum_{p=0}^{n-1} \frac{(-1)^p . (2n-2p-1)}{(p+1)(2n-p)} P_{n-p-1}^m \right]$$

$$= -\frac{1}{2n+1} . \frac{\partial}{\partial m} P_n^m + \sum_{p=0}^{\infty} \frac{2n+2p+3}{(p+1)(2n+p+2)} \frac{\partial}{\partial m} . P_{n+p+1}^m$$

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$$+\sum_{p=0}^{n-1}\frac{(-1)^p(2n-2p-1)}{(p+1)\,2n-p)}\frac{\partial}{\partial m}.\,P^m_{n-p-1}.$$

Here for

$$\frac{\partial}{\partial m}P^m_n,\quad \frac{\partial}{\partial m}P^m_{n+p+1} \text{ and } \frac{\partial}{\partial m}P^m_{n-p-1}$$

we have to substitute in terms of associated Legendre functions from (3.1) or (4.1) thus obtaining double series.

The general case of

$$\frac{\partial^{\,p+q}}{\partial m^p.\ \partial n^q}\left\{r^n.\,P^m_n.\begin{matrix}\cos\\ \sin\end{matrix}\,m\phi\right\}$$

can be treated similarly by using the results of § 3 and § 4 in the results of § 2. But it is easy to see that the expressions will be tremendously complicated giving a number of double series.

In conclusion I wish to express my best thanks to Prof. Ganesh Prasad for kindly suggesting to me this problem and for the interest he has taken in this paper.

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Chapter III

The k_n-function

Paper I

~~II~~

On Some Results Involving the k-Function, a Particular Case of the Confluent Hypergeometric Function

BY

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(Read, the 3rd July, 1932.)

Introduction :—In papers * recently published, Prof. H. Bateman has developed the theory of the function $k_n(x)$, which is defined by him for all real values of n and x by the integral

$$\frac{2}{\pi} \cdot \int_0^{\frac{\pi}{2}} \cos\,(x \tan\,\theta - n\theta) d\theta.$$

* 1. " On the k-function, a particular case of the confluent hypergeometric function "—the *Trans. of the American Math. Society,* Vol. 33, No. 4, pp. 817-831.

2. " Solutions of a certain partial differential equation "—the *Proceedings of the National Academy of Sciences*, Vol. 17, No. 10, pp. 562-567, October, 1931.

3. " Relations between confluent hypergeometric functions "—the *Prcc. National Academy of Sciences*, Vol. 17, No. 12, pp. 689-690, December, 1931.

These three papers will be referred to here as B_1, B_2, and B_3. Prof. Bateman writes to me—" I shall be glad if you will kindly correct the mistake in sign relating to $k_1(x)$, if you publish your work."—I take this opportunity to correct a slip in B_1. On page 818, line 7 from the top, read + sign before the integral $\frac{2x}{\pi}\int_0^\infty \frac{\cos\,(xt).\;dt}{(1+t^2)^{\frac{1}{2}}}$ in place of the − ve sign and read + sign before $K_0(x)$ in line 8 instead of the − ve sign. Prof. Bateman uses the definition of $K_n(x)$ used in Watson's *Bessel Functions.* Similarly " the reference to Szegö on p. 825 should be to the paper (*Math. Zeit.* Vol. 25) cited on p. 829 "—as he writes to me. Also on p. 826, for m$\geq$s, value of $\int_0^\infty k_{2s}(x)\,\left(\frac{dx}{x}\right) e^{-x}.L_m(2x)$ should be $\frac{(-1)^{s-1}}{s}$ and not $\frac{1}{s}$. This correction has been suggested by Prof. Bateman himself when the error was brought to his notice by me.

Another definition, $k_n(x)$

$$= \frac{2}{\pi} \sum_{m=-\infty}^{\infty} \frac{\sin (2m-n)\frac{\pi}{2}}{2m-n} k_{2m}(x),$$

depending upon cardinal functions and interpolation formulæ is also given in the same papers and the theory is developed from the consideration of both of these definitions. This function is a particular case of the confluent hypergeometric function $W_{k,m}(x)$ and in fact *

$$k_{2n}(x)\, \Gamma(1+n) = W_{n,\frac{1}{2}}(2x),\ x>0 \Big\}.$$

This function arises as a solution of the differential equation $xy''=(x-n)y$, which plays an important part in the recent researches of Kármán † and Tollmien ‡ on the turbulent motion of a fluid. This equation was submitted by Kármán to Bateman for further investigation. In view of the importance of the function in physical problems as well as from purely mathematical aspect, the following short study of its properties seems to be justifiable.⊕

In art. 1 we obtain some integrals for $k_n(x)$. In art. 2, we study the addition theorem given by Bateman without proof in B_2. In art. 3, an expression for $k_n(x)$, n an odd integer, in terms of Bessel functions $K_n(x)$ is derived and this is made use of in obtaining certain definite integrals involving $k_n(x)$ with n an odd integer. In art 4 we obtain other definite integrals involving $k_n(x)$ and in art. 5, some miscellaneous results are derived. Many of the results given below are believed to be new.

§ 1

A contour integral for $k_{2n}(x)$, n being a +ve integer.§

$$(1.1)\quad k_{2n}(x) = \frac{1}{2\pi i}\int^{(0+)} e^{x\left\{\frac{t^2-1}{t^2+1}\right\}} t^{-2n-1}\,dt\,;$$

the integration being taken round any contour encircling the origin once counter-clock-wise.

* See B_3 p. 689. † *Göttinger Nachrichten*, Vol. 30.

‡ *Göttinger Nachrichten*, 1929, " Über die Enstelung der Turbulenz."

§ As mentioned to the author by Prof. Bateman, this contour integral is closely related to the integral (4) on p. 563 of B_2 and by formulae (1) and (5) of B_3 this integral is expressible in terms $k_{2n}(x)$ when $v=-2n$ when n is zero or a + ve integer.

⊕ Some properties of this function have also been developed by (1) M. Lerch (Crelle's Journal, 1905. Vol 130, pp 47-65), who writes

$$e^{-\frac{\omega}{1-z}} = \sum_{\mu=0}^{\infty} \psi_\mu(\omega)\, z^\mu$$

(2) Theodor Sexl: (Zeit. f. Ph. 56, 1929, p72) " Zur Theorie der bei der Wellenmechanischen Behandlung des radioaktiven α-Zerfalls auftretenden Differentialgleichungen"

The differential equation considered by Sexl is

$$y'' + \left(1-\frac{k}{x}\right)y = 0$$

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Proof :

According to Bateman's definition

$$k_{2n}(x)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos\,(2n\theta-x\tan\theta)d\theta$$

The above

$$=\frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cos\,(2n\theta-x\tan\theta)\,d\theta+\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\cos\,(2n\theta-x\tan\theta)\,d\theta$$

$$=\frac{1}{\pi}\int_0^{\pi} \cos\,(2n\theta-x\tan\theta)\,d\theta \text{ (putting } \pi-\theta \text{ for } \theta \text{ in the 2nd integral)}$$

$$=\frac{1}{2\pi}\int_0^{\pi} e^{i(2n\theta-x\tan\theta)}\,d\theta+\frac{1}{2\pi}\int_0^{\pi} e^{-i(2n\theta-x\tan\theta)}\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{\pi} e^{i(2n\theta-x\tan\theta)}\,d\theta \text{ (putting} -\theta \text{ for } \theta \text{ in the 2nd integral)}$$

$$=-\frac{1}{2\pi i}\int e^{x\left\{\frac{t^2-1}{t^2+1}\right\}}\cdot\frac{dt}{t^{2n+1}} \text{ (putting } t=e^{-\theta i}\text{)}$$

the integration with respect to t being carried out round the circle $|\,t\,|=1$ of unit radius from $\theta=\pi$ to $\theta=-\pi$, *i.e.*, in the clock-wise direction.

Thus, dropping the minus sign, $k_{2n}(x)=\int^{(0+)} e^{x\left\{\frac{t^2-1}{t^2+1}\right\}}\,\frac{dt}{t^{2n+1}}$

the integration being carried round the circle in a counter-clock-wise direction.

An integral for $k_n(x)$ for all real values of n and x.

$$(1.11)\quad k_n(x)=\frac{1}{\pi i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{x\left\{\frac{t^2-1}{t^2+1}\right\}}\,\frac{dt}{t^{n+1}},\ (t=e^{-\theta i})$$

the integration being carried round a semi-circle of unit radius from

$\theta=\frac{\pi}{2}$ to $\theta=-\frac{\pi}{2}$ in the counter-clock-wise direction.

Proof :

For all real values of n and x

$$k_n(x)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos\,[n\theta-x\tan\theta]d\theta$$

$$=\frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{i(n\theta-x\tan\theta)}d\theta+\frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-i(n\theta-x\tan\theta)}d\theta$$

$$=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(n\theta-x\tan\theta)}\,d\theta \text{ (putting } -\theta \text{ for } \theta \text{ in the second integral)}$$

$$=-\frac{1}{\pi i}\int e^{x\left\{\frac{t^2-1}{t^2+1}\right\}}\frac{dt}{t^{n+1}} \text{ (putting } t=e^{-\theta i}\text{)},$$

the integral being carried out with respect to t round the semi-circle

of unit radius from $\theta=\frac{\pi}{2}$ to $\theta=-\frac{\pi}{2}$. Thus dropping the minus sign,

$$k_n(x)=\frac{1}{\pi i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{x\left\{\frac{t^2-1}{t^2+1}\right\}}.\frac{dt}{t^{n+1}},\ (t=e^{-\theta i})$$

integral being carried round the semi-circle in a counter-clock-wise direction.

An integral for $k_{2n}(x)$ for + ve real values of x.

$$\Gamma(1+n)\ \ k_{2n}(x)=W_{n,\frac{1}{2}}(2x) \text{ for } x>0$$

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But

$$*W_{k,m}(z) = -\frac{1}{2\pi i}\,\Gamma(k+\tfrac{1}{2}-m)e^{-\frac{1}{2}z}.z^k$$

$$\int_{\infty}^{(0+)} (-t)^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m} e^{-t}\,dt$$

where the arg z has its principal value and the contour is so chosen that the point $t=-z$ is outside it. The integrand is rendered one-valued by taking $|\arg(t)| \leq \pi$ and taking the value of $\arg\left(1+\frac{t}{z}\right)$ which tends to zero as $t \longrightarrow 0$ by a path lying inside the contour. This formula becomes nugatory when $k-\frac{1}{2}-m$ is a $-ve$ integer. To over-come this difficulty we observe that wherever $R(k-\frac{1}{2}-m) \leq 0$ and $k-\frac{1}{2}-m$ is not an integer we transform the contour integral into an infinite integral and so when $R(k-\frac{1}{2}-m) \leq 0$,

$$W_{k,m}(z) = \frac{e^{-\frac{1}{2}z}.\ z^k}{\Gamma(\frac{1}{2}-k+m)}\int_0^{\infty} t^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m}.\ e^{-t}\,dt$$

This formula suffices to define $W_{k,m}(z)$ in the critical cases where $m+\frac{1}{2}-k$ is $a+ve$ integer and so $W_{k,m}(z)$ is defined for all values of k and m and all values of z except $-ve$ real values.

Hence we get for n not a $-ve$ integer,

$$(1.2)\quad k_{2n}(x) = -\frac{\Gamma n}{2\pi i\ \Gamma(1+n)}.\ e^{-x}.\ x^k.\int_{\infty}^{(0+)} (-t)^{-n}(t+2x)^n e^{-t}\,dt$$

$$= -\frac{1}{2\pi i.\ n}\,e^{-x}.\ x^k.\ \int_{\infty}^{(0+)} e^{-t}(-t)^{-n}(t+2x)^n\,dt$$

the contour being so chosen that the point $t=-2x$ is outside it. The integrand is made one valued by taking $|\arg(t)| \leq \pi$ and taking value of $\left|\arg\left(\frac{t}{2x}+1\right)\right|$ which tends to zero with t.

* Whittaker and Watson, *Modern Analysis*, 3rd Edition, pp. 339-340.

When $n-1\leq 0$

$$(1.21)\quad k_{2n}(x)=\frac{e^{-x}.\,x^k}{\Gamma(1-n)\Gamma(1+n)}.\int_0^\infty t^{-n}.\,(2x+t)^n.\,e^{-t}.\,dt$$

$$=\frac{e^{-x}.\,x^k.\sin n\pi}{n\pi}.\int_0^\infty t^{-n}(2x+t)^n.\,e^{-t}.\,dt$$

This suffices to define $k_{2n}(x)$ when $1-n$ is $a+ve$ integer.

Contour integral of Barnes' type for $k_n(x)$.

$$*\mathrm{W}_{k,m}(z)=\frac{e^{-\frac{1}{2}z}.z^k}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma(s)\;\Gamma(-s-k-m+\frac{1}{2})\Gamma(-s-k-m+\frac{1}{2})}{\Gamma(-k-m+\frac{1}{2})\Gamma(-k+m+\frac{1}{2})}z^s ds$$

for all values of z such that $|\arg z|<\pi$ and for all values of arg z such that $\pi\leq|\arg z|<\frac{3}{2}\pi$ and such that k and $k-1$ are not + ve integers.

Therefore, remembering that $k_{2n}(x)=\frac{1}{\Gamma(1+n)}\mathrm{W}_{n,\frac{1}{2}}(2x)$ we have

for values of $\left.\begin{array}{l} n>0 \\ \text{and} \\ x>0\end{array}\right\}$ such that n or $n-1$ is not a + ve integer

$$(1.3)\quad k_{2n}(x)=\frac{1}{\Gamma(1+n)}\frac{e^{-x}(2x)^n}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma(s)\Gamma(-s-n)\;\Gamma(-s-n-1)}{\Gamma(-n)\Gamma(-n+1)\;(2x)^s}ds,$$

§ 2

Addition theorem for all real values of n, for $k_n(x)$ as stated by Bateman in B_2 p. 567 without proof is as follows ;

$$(2.1)\quad k_n(x+y)=k_n(x).\;k_0(y)+k_{n-2}(x).\;k_2(y)+..,x>0,\;y\geq 0.$$

Addition theorem for $k_{2n}(x)$, *n being a + ve integer.*

$$(2.11)\quad k_{2n}(x+y)=\sum_{m=-\infty}^{\infty}k_{2m}(y).\;k_{2n-2m}(x)$$

* Whittaker and Watson, *Modern Analysis*, pp. 343-345.

Proof :

$$k_{2n}(x+y)= \frac{1}{2\pi i}\int^{(0+)} t^{-2n-1}.\ e^{(x+y)\left\{\frac{t^2-1}{t^2+1}\right\}}.\,dt$$

by using (1.1)

$$=\frac{1}{2\pi i}\int^{(0+)} \left\{ \sum_{m=-\infty}^{\infty} t^{2m-2n-1}.k_{2m}(y).\ e^{x\left\{\frac{t^2-1}{t^2+1}\right\}}\ dt \right\}$$

$$\left[\text{as } e^{y\left(\frac{t^2-1}{t^2+1}\right)}=\sum_{m=-\infty}^{\infty} k_{2m}(y).\ t^{2m}\right]^*$$

$$=\frac{1}{2\pi i}\sum_{m=-\infty}^{\infty} k_{2n}(y)\int^{(0+)} t^{2m-2n-1}.\ e^{x\left\{\frac{t^2-1}{t^2+1}\right\}}\ dt$$

$$=\sum_{m=-\infty}^{\infty} k_{2m}(y)k_{2n-2m}(x).$$

Proof of

$$(2.12)\quad k_n(x+y)=\sum_{m=-\infty}^{\infty} k_{2m}(y)k_{2n-2m}(x)$$

for all real values of n and $y>0$.

$$k_n(x+y)=\frac{1}{\pi i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t^{-n-1}.\ e^{(x+y)\left\{\frac{t^2-1}{t^2+1}\right\}}.\,dt$$

by using (1.11)

$$=\frac{1}{\pi i}\int \left\{\sum_{m=-\infty}^{\infty} t^{2m-n-1}.\ k_{2m}(y)e^{x\left\{\frac{t^2-1}{t^2+1}\right\}}\right\}dt,$$

$y>0$

* B_2, pp. 818-819.

$$\left[\text{as } e^{y\left\{\frac{t^2-1}{t^2+1}\right\}}=\sum_{m=-\infty}^{\infty} k_{2m}(y).\ t^{2m}\right]$$

$$=\frac{1}{\pi i}\sum_{m=-\infty}^{\infty} k_{2m}(y)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t^{2m-n-1}.\ e^{x\left\{\frac{t^2-1}{t^2+1}\right\}}.\ at$$

$$=\sum_{m=-\infty}^{\infty} k_{2m}(y).\ k_{n-2m}(x)$$

Proof of the addition theorem by means of Parseval's theorem:

$$e^{iy\tan\theta}=k_0(y)+(\cos 2\theta+i\sin 2\theta)k_2(y)+\dots,\ y\geq 0$$

$$e^{ix\tan\theta}.\ e^{-ni\theta}=\sum_{m=0}^{\infty} k_{n-2m}(x)\ (\cos 2m\theta-i\sin 2m\theta),\ x>0.$$

$\therefore$ By Parseval's theorem,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} e^{i(x+y)\tan\theta-n\theta i}.\ d\theta$$

$$=2k_n(x).\ k_0(y)+2\sum_{m=1}^{\infty} k_{n-2m}(x)k_{2m}(y)$$

$$=2k_n(x+y)$$

or $k_n(x+y)=k_0(y).\ k_n(x)+\sum_{m=1}^{\infty} k_{n-2m}(x).\ k_{2m}(y)$

§ 3

An expression for $k_n(x)$, n being an odd + ve integer in terms of Bessel functions.

If n is an odd integer

$$\cos n\theta=\tfrac{1}{2}\Big[(2\cos\theta)^n-n(2\cos\theta)^{n-2}+\frac{n(n-3)}{1.2}(2\cos\theta)^{n-4}$$

$$-\frac{n(n-4)(n-5)}{1.2.3}(2\cos\theta)^{n-6}+\dots$$

$$(-1)^{\frac{n-1}{2}}.\ n,\ (2\cos\theta)\Big]$$

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$$= \tfrac{1}{2}\left[\frac{2^n}{(1+t^2)^{\frac{n}{2}}} - \frac{n.\,2^{n-2}}{(1+t^2)^{\frac{n}{2}-1}} + \frac{n(n-3)}{1.2}\cdot\frac{2^{n-4}}{(1+t^2)^{\frac{n}{2}-2}}\right.$$

$$\left.(-1)^{\frac{n-1}{2}}.\frac{n.2}{(1+t^2)^{\frac{1}{2}}}\right] \text{ if } t=\tan\theta\text{...(I)}$$

and $\sin n\theta=\sin\theta\,[(2\cos\theta)^{n-1}-(n-2)(2\cos\theta)^{n-3}$

$$+\frac{(n-3)(n-4)}{1.2}.\ (2\cos\theta)_{+}^{n-5}...(-1)^{\frac{n-1}{2}}]$$

$$=\frac{t}{(1+t^2)^{\frac{1}{2}}}\left[\frac{2^{n-1}}{(1+t^2)^{\frac{n-1}{2}}}-\frac{(n-2)2^{n-3}}{(1+t^2)^{\frac{n-3}{2}}}+...(-1)^{\frac{n-1}{2}}\right]$$

$$\text{if } t=\tan\theta\text{...(II)}$$

Again,

$$\int_0^\infty\frac{\cos xt}{(1+t^2)^{m+\frac{1}{2}}}\,dt=\frac{x^m.\,\pi^{\frac{1}{2}}}{2^m.\,\Gamma(m+\frac{1}{2})}\,K_m(x),$$

taking m to be a+ve integer or zero and using Bassets'* notation for $K_m(x)$.

Now,

$$k_n(x)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos(x\tan\theta-n\theta)d\theta,$$

n, a+ve integer and $x>0$ and real

$$=\frac{2}{\pi}\int_0^\infty\frac{\cos xt}{(1+t^2)}\,dt\left[\tfrac{1}{2}\left\{\frac{2^n}{(1+t^2)^{\frac{n}{2}}}-\frac{n.\,2^{n-2}}{(1+t^2)^{\frac{n}{2}-1}}+...\right.\right.$$

$$\left.(-1)^{\frac{n-1}{2}}.\frac{2}{(1+t^2)^{\frac{1}{2}}}\right]$$

* See the foot-note on page 373, Whittaker and Watson, *Modern Analysis*, 3rd edition.

$$+\frac{2}{\pi}\int_0^\infty \frac{\sin xt}{(1+t^2)}dt\left[\frac{t}{(1+t^2)^{\frac{1}{2}}}\left\{\frac{2^{n-1}}{(1+t^2)^{\frac{n-1}{2}}}-\frac{(n-2).\,2^{n-3}}{(1+t^2)^{\frac{n-3}{2}}}+\ldots+(-1)^{\frac{n-1}{2}}\right\}\right]$$

(putting $t=\tan\theta$ and using I and II)

$$=\frac{2}{\pi}\int_0^\infty \cos xt\,dt\left[\frac{2^{n-1}}{(1+t^2)^{\frac{n}{2}+1}}-\frac{n.\,2^{n-3}}{(1+t^2)^{\frac{n}{2}}}+\ldots(-1)^{\frac{n-1}{2}}\quad\frac{n.\,2}{(1+t^2)^{\frac{3}{2}}}\right.$$

$$+\frac{2x}{\pi}\int_0^\infty \cos xt.\,dt\left[\frac{2^{n-1}}{n(1+t^2)^{\frac{n}{2}}}-\frac{(n-2).\,2^{n-3}}{(n-2)\,(1+t^2)^{\frac{n}{2}-1}}\right.$$

$$\left.+\frac{(n-3)(n-4)2^{n-5}}{1.\,2.\,(n-4)(t^2+1)^{\frac{n}{2}-2}}+\ldots(-1)^{\frac{n-1}{2}}\quad\frac{1}{(1+t^2)^{\frac{1}{2}}}\right]$$

(integrating the second integral by parts)

$$=\frac{2}{\pi}.\,\pi^{\frac{1}{2}}\left[\frac{2^{n-1}.\quad x^{\frac{n+1}{2}}}{2^{\frac{n+1}{2}}\,\Gamma(\frac{n}{2}+1)}\,K_{\frac{n+1}{2}}(x)-\frac{n.\,2^{n-3}.\,x^{\frac{n-1}{2}}\,K_{\frac{n-1}{2}}(x)}{2^{\frac{n-1}{2}}.\;\Gamma(\frac{n}{2}+1)}\right.$$

$$\left.+\frac{n(n-3)}{1.\,2}\cdot\frac{2^{n-3}.\,x^{\frac{n-3}{2}}.\,K_{\frac{n-3}{2}}(x)}{2^{\frac{n-3}{2}}.\;\Gamma(\frac{n}{2}-1)}+\ldots+\frac{(-1)^{\frac{n-1}{2}}.\,x\,K_1(x)}{2\Gamma\frac{3}{2}}\right]$$

$$+\frac{2x}{\pi}.\,\pi^{\frac{1}{2}}\left[\frac{2^{n-1}.\,x^{\frac{n-1}{2}}.\,K_{\frac{n-1}{2}}(x)}{n.\,2^{n-\frac{1}{2}},\;\Gamma\frac{n}{2}}-\frac{2^{n-3}.\,x^{\frac{n-3}{2}}.\,K_{\frac{n-3}{2}}(x)}{2^{\frac{n-3}{2}}.\;\Gamma(\frac{n}{2}-1)}\right.$$

$$\left.+\ldots\frac{(-1)^{\frac{n-1}{2}}K_0(x)}{\Gamma\frac{1}{2}}\right]$$

$$=\frac{2}{\pi^{\frac{1}{2}}}\left[\frac{2^{\frac{n-3}{2}}.\,x^{\frac{n-1}{2}}.\,K_{\frac{n+1}{2}}(x)}{\Gamma(\frac{n}{2}+1)}-\frac{n.\,2^{\frac{n-5}{2}}.\,x^{\frac{n-1}{2}}.\,K_{\frac{n-1}{2}}(x)'}{\Gamma\frac{n}{2}}+\ldots\right.$$

$$\left.+\frac{(-1)^{\frac{n-1}{2}}.\,xK_1(x)}{2.\,\Gamma\frac{3}{2}}\right]$$

$$+\frac{2x}{\pi^{\frac{1}{2}}}\left[\frac{2^{\frac{n-1}{2}}\cdot x^{\frac{n-1}{2}}\cdot K_{\frac{n+1}{2}}(x)}{n.\,\Gamma\frac{n}{2}}-\frac{2^{\frac{n-3}{2}}\cdot(x)^{\frac{n-3}{2}}\cdot}{\Gamma(\frac{n}{2}-1)}K_{\frac{n-3}{2}}+\dots\right.$$

$$\left.+(-1)^{\frac{n-1}{2}}\cdot\frac{K_0(x)}{\Gamma\frac{1}{2}}\right]$$

By means of the recurrence formulae for $K_n(x)$, this formula can be reduced in terms of $K_0(x)$ and $K_1(x)$ and thus when n is an odd integer $k_n(x)$ can be expressed in terms of $K_0(x)$ and $K_1(x)$ as stated by Bateman on page 818 of B_1.

When n is an odd integer and $R(\mu+1)>0$

(3.2) $$I=\int_0^\infty k_n(x).\,x^{\mu}.\,dx$$

$$=\frac{2}{\pi^{\frac{1}{2}}}\left[\frac{2^{\frac{n-3}{2}}}{\Gamma(\frac{n}{2}+1)}\cdot\int_0^\infty K_{\frac{n+1}{2}}(x).x^{\mu+\frac{n+1}{2}}dx\right.$$

$$-\frac{n.\,2^{\frac{n-5}{2}}}{\Gamma\frac{n}{2}}\cdot\int_0^\infty K_{\frac{n-1}{2}}(x).\,x^{\mu+\frac{n-1}{2}}dx+\dots$$

$$\left.+\frac{(-1)^{\frac{n-1}{2}}}{2\Gamma\frac{3}{2}}\int_0^\infty x^{\mu+1}.\,K_1(x)dx\right]$$

$$+\frac{2}{\pi^{\frac{1}{2}}}\left[\frac{2^{\frac{n-1}{2}}}{n\Gamma\frac{n}{2}}\cdot\int_0^\infty K_{\frac{n-1}{2}}(x).\,x^{\frac{n-1}{2}+\mu+1}dx\right.$$

$$-\frac{2^{\frac{n-3}{2}}}{\Gamma(\frac{n}{2}-1)}\cdot\int_0^\infty x^{\frac{n-3}{2}+\mu+1}\;K_{\frac{n-3}{2}}(x)dx+\dots$$

$$\left.+(-1)^{\frac{n-1}{2}}\cdot\int_0^\infty\frac{K_0(x)x^{\mu+1}dx}{\Gamma\frac{1}{2}}\right]$$

making use of (3.1)

$$=\frac{2}{\pi^{\frac{1}{2}}}\left[\frac{2^{\frac{n-3}{2}}}{\Gamma(\frac{n}{2}+1)}\left\{2^{\mu+\frac{n-1}{2}}\cdot\Gamma\frac{\mu+1}{2}\times\Gamma\frac{\mu+n+2}{2}\right\}\right.$$

$$-\frac{n.2^{\frac{n-3}{2}}}{\Gamma\frac{n}{2}}\Big\{2^{\mu+\frac{n-3}{2}}\Gamma\frac{\mu+1}{2}\Gamma\frac{\mu+n}{2}+\ldots$$

$$+\frac{(-1)^{\frac{n-1}{2}}}{2\Gamma\frac{3}{2}}2^{\mu}.\Gamma\frac{\mu+1}{2}\Gamma\frac{\mu+3}{2}\Big\}\Big]$$

$$+\frac{2}{\pi^{\frac{1}{2}}}\Big[\frac{2^{\frac{n-1}{2}}}{n\Gamma\frac{n}{2}}\Big\{2^{\mu+\frac{n-1}{2}}.\Gamma\frac{\mu+2}{2}.\Gamma\frac{\mu+n+1}{2}\Big\}$$

$$-\frac{2^{\frac{n-3}{2}}}{\Gamma(\frac{n}{2}-1)}\Big\{2^{\mu+\frac{n-3}{2}}\Gamma\frac{\mu+2}{2}\times\Gamma\frac{\mu+n-1}{2}\Big\}\Big]+\ldots$$

$$+(-1)^{\frac{n-1}{2}}.\frac{2^{\mu}}{\Gamma\frac{1}{2}}.\Gamma\frac{\mu+2}{2}\Gamma\frac{\mu+2}{2}\Big\}\Big]$$

making use of the formula *

$$\int_0^\infty K_0(t).\ t^{\mu-1}\,dt=2^{\mu-2}.\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\mu+\nu}{2}\right),\ R(\mu)>R(\nu);$$

$$=\frac{2}{\pi^{\frac{1}{2}}}\Big\{\Gamma\frac{\mu+1}{2}\Big[\frac{2^{\mu+n-2}.\Gamma\frac{\mu+n+2}{2}}{\Gamma(\frac{n}{2}+1)}-\frac{n.2^{n+\mu-4}}{\Gamma\frac{n}{2}}\Gamma\frac{\mu+n}{2}+\ldots$$

$$+\frac{(-)^{\frac{n-1}{2}}.2^{\mu}.\Gamma\frac{\mu+3}{2}}{2\Gamma\frac{3}{2}}\Big]$$

$$+\Gamma\left(\frac{\mu}{2}+1\right)\Big[\frac{2^{\mu+n-1}.\Gamma\frac{\mu+n+1}{2}}{n\Gamma\frac{n}{2}}-\frac{2^{\mu+n-3}}{\Gamma(\frac{n}{2}-1)}.\Gamma\frac{\mu+n-1}{2}+\ldots$$

$$+\frac{(-1)^{\frac{n-1}{2}}.2^{\mu}}{\Gamma\frac{1}{2}}.\Gamma\frac{\mu+2}{2}\Big]\Big\}$$

* See Watson's *The theory of Bessel Functions.*

n being an odd integer we have

$$(3.21)\quad \mathrm{I} = \int_0^\infty k_n(x).\, \mathrm{K}_\mu(x).\, x^{\rho-1}.\, dx$$

$$= \frac{2}{\pi^{\frac{1}{2}}} \cdot \Bigg[\frac{2^{\frac{n-3}{2}}}{\Gamma(\frac{n}{2}+1)} \cdot \int_0^\infty \mathrm{K}_{\frac{n+1}{2}}(x).\, \mathrm{K}_\mu(x).\, x^{\rho-1+\frac{n+1}{2}} dx$$

$$- \frac{n.\, 2^{\frac{n-5}{2}}}{\Gamma\frac{n}{2}} \cdot \int_0^\infty \mathrm{K}_{\frac{n-1}{2}}(x).\, \mathrm{K}_\mu(x).\, x^{\rho-1+\frac{n-1}{2}}.\, dx + \ldots$$

$$+ \frac{(-1)^{\frac{n-1}{2}}}{2\Gamma\frac{3}{2}} \cdot \int_0^\infty x^\rho.\, \mathrm{K}_1.\, \mathrm{K}_\mu.\, dx \Bigg]$$

$$+ \frac{2}{\pi^{\frac{1}{2}}} \Bigg[\frac{2^{\frac{n-1}{2}}}{n\Gamma\frac{n}{2}} \cdot \int_0^\infty \mathrm{K}_{\frac{n-1}{2}}.\, x^{\frac{n-1}{2}+\rho}.\, \mathrm{K}_\mu.\, dx$$

$$- \frac{2^{\frac{n-3}{2}}}{\Gamma(\frac{n}{2}-1)} \cdot \int_0^\infty x^{\frac{n-3}{2}+\rho}.\, \mathrm{K}_{\frac{n-3}{2}}.\, \mathrm{K}_\mu.\, dx \text{——} +$$

$$\ldots\ldots + (-1)^{\frac{n-1}{2}} \int_0^\infty \frac{\mathrm{K}_0 \mathrm{K}_\mu.\, x\, dx}{\Gamma\frac{1}{2}} \Bigg]$$

making use of (3.1)

$$= \frac{2}{\pi^{\frac{1}{2}}} \Bigg[\frac{2^{\frac{n-3}{2}}}{\Gamma(\frac{n}{2}+1)} \cdot \frac{2^{\frac{n+1}{2}+\rho-3}}{\Gamma(\rho+\frac{n+1}{2})} \Gamma\frac{\rho+n+1+\mu}{2}$$

$$\cdot\ \Gamma\frac{\rho+n+1-\mu}{2} \Gamma\frac{\rho-\mu}{2} \Gamma\frac{\rho+\mu}{2}$$

$$- \frac{n}{\Gamma\frac{n}{2}} \cdot \frac{2^{\frac{n-5}{2}}.\, 2^{\rho+\frac{n-1}{2}-3}}{\Gamma(\rho+\frac{n-1}{2})} \Gamma\frac{\rho+n-1-\mu}{2} \Gamma\frac{\rho+n-1+\mu}{2} \times$$

$$\Gamma\frac{\rho-\mu}{2} \Gamma\frac{\rho+\mu}{2} + \ldots \frac{(-1)^{\frac{n-1}{2}}}{2\Gamma\frac{3}{2}} \frac{2^{\rho-2}}{\Gamma(\rho+1)} \times$$

$$\Gamma\frac{\rho+2-\mu}{2}\, \Gamma\frac{\rho+2+\mu}{2}\, \Gamma\frac{\rho+\mu}{2} \Gamma\frac{\rho-\mu}{2}$$

$$+\frac{2}{\pi^{\frac{1}{2}}}.\Bigg[\frac{2^{\frac{n-1}{2}}.\,2^{\frac{n+1}{2}+\rho-3}}{n\Gamma\frac{n}{2}\;\Gamma\left(\rho+\frac{n+1}{2}\right)}.\,\Gamma\frac{\rho+n-\mu}{2}\,.\,\Gamma\frac{\rho+n+\mu}{2}\times$$

$$\Gamma\frac{\rho-1-\mu}{2}\;\Gamma\frac{\rho-1+\mu}{2}$$

$$-\frac{2^{\frac{n-3}{2}}}{\Gamma\left(\frac{n}{2}-1\right)}.\frac{2^{\frac{n-1}{2}+\rho-3}}{\Gamma\,\rho+\frac{n-1}{2}}\,\Gamma\left(\frac{\rho+n-2-\mu}{2}\right)\times$$

$$\Gamma\frac{\rho+n-2+\mu}{2}\;\Gamma\frac{\rho-1-\mu}{2}\;\Gamma\frac{\rho-1+\mu}{2}\;+\ldots$$

$$+\frac{(-1)^{\frac{n-1}{2}}.\,2^{\rho-2}}{\Gamma\frac{1}{2}\;\Gamma(\rho+1)}\,\Gamma\frac{\rho+1-\mu}{2}\,\Gamma\frac{\rho+1+\mu}{2}\times$$

$$\Gamma\frac{\rho-1+\mu}{2}\;\Gamma\frac{\rho-1+\mu}{2}\;\Bigg]$$

making use of the integral *

$$\int_0^\infty K_\lambda(x)K_\mu(x).\,x^{\rho-1}.\,dx$$

$$\frac{2^{\rho-3}}{\Gamma\rho}.\,\Gamma\frac{\rho+\lambda-\mu}{2}\,.\,\Gamma\frac{\rho+\lambda+\mu}{2}\,.\,\Gamma\frac{\rho-\lambda+\mu}{2}\,.\,\Gamma\frac{\rho-\lambda-\mu}{2}$$

$$=\frac{2}{\pi^{\frac{1}{2}}}\Bigg[\Bigg\{\frac{2^{n+\rho-4}}{\Gamma\left(\frac{n}{2}+1\right)\;\Gamma\left(\rho+\frac{n+1}{2}\right)}\Gamma\frac{\rho+n+1-\mu}{2}.\,\Gamma\frac{\rho+n+1+\mu}{2}$$

$$-\frac{n.\;\;2^{\rho+n-6}}{\Gamma\frac{n}{2}\;\Gamma\left(\rho+\frac{n-1}{2}\right)}.\,\Gamma\frac{\rho+n-1-\mu}{2}\Gamma\frac{\rho+n-1+\mu}{2}+\ldots$$

$$+\frac{(-1)^{\frac{n-1}{2}}.\,2^{\rho-2}}{\Gamma\frac{3}{2}}.\,\Gamma\frac{\rho+2-\mu}{2}\Gamma\frac{\rho+2+\mu}{2}\Bigg\}\Gamma\frac{\rho+\mu}{2}\Gamma\frac{\rho-\mu}{2}$$

$$+\Gamma\frac{\rho-1-\mu}{2}\;\Gamma\frac{\rho-1+\mu}{2}\;\Bigg\{\frac{2^{n+\rho-3}}{n.\;\Gamma\frac{n}{2}\;\Gamma\left(\rho+\frac{n+1}{2}\right)}\times$$

* Titschmarch, *Journal of the L. M. Society*, Vol. 2, 1927.

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$$\times \Gamma\frac{\rho+n-\mu}{2}.\,\Gamma\frac{\rho+n+\mu}{2}$$

$$+\frac{2^{\rho+n-5}}{\Gamma(\frac{n}{2}-1)\,\Gamma(\rho+\frac{n-1}{2})}\Gamma\frac{n+\rho-2-\mu}{2}.\,\Gamma\frac{\rho+n-2+\mu}{2}+\ldots$$

$$+\frac{(-1)^{\frac{n-1}{2}}}{\Gamma\frac{1}{2}\,\Gamma(\rho+1)}\Gamma\frac{\rho+1-\mu}{2}\,\Gamma\frac{\rho+1+\mu}{2}\Big\}\Big].$$

When R $(\cosh\alpha)>-1$ and n an odd integer we have

$$(3.22)\quad \mathrm{I}=\int_0^\infty e^{-\cosh\alpha}.\,k_n(x).\,x^\mu\,dx.$$

$$=2^{\frac{1}{2}}\Big[\Gamma(\mu+1)\Big\{\frac{2^{\frac{n-3}{2}}}{\Gamma(\frac{n}{2}+1)}\frac{\Gamma(\mu+n+2)\;\mathrm{P}_{\frac{n}{2}}^{-\mu-\frac{n}{2}-1}(\cosh\alpha)}{\sinh_\alpha^{\mu+\frac{n}{2}+1}}$$

$$-\frac{n.\,2^{\frac{n-5}{2}}}{\Gamma\frac{n}{2}}\frac{\Gamma(\mu+n)\,\mathrm{P}_{\frac{n}{2}-1}^{-\mu-\frac{n}{2}}(\cosh\alpha)}{\sinh^{\mu+\frac{n}{2}}\alpha}+\ldots$$

$$+\frac{(-1)^{\frac{n-1}{2}}}{2\Gamma\frac{3}{2}}.\,\Gamma(\mu+3).\,\frac{\mathrm{P}_{\frac{1}{2}}^{-\mu-\frac{3}{2}}(\cosh\alpha)}{\sinh^{\mu+\frac{3}{2}}\alpha}\Big\}\Big]$$

$$+\Gamma(\mu+2)\Big\{\frac{2^{\frac{n-1}{2}}}{n\Gamma\frac{n}{2}}.\,\frac{\Gamma\,\mu+n+1\,.\,\mathrm{P}_{\frac{n}{2}-1}^{-\mu-\frac{n}{2}-1}(\cosh\alpha)}{\sinh^{\mu+\frac{n}{2}+1}_{\alpha}}$$

$$-\frac{2^{\frac{n-3}{2}}}{\Gamma(\frac{n}{2}-1)}\frac{\Gamma(\mu+n-1)\,\mathrm{P}_{\frac{n}{2}-2}^{-\mu-\frac{n}{2}}(\cosh\alpha)}{\sinh^{\mu+\frac{n}{2}}\alpha}+\ldots$$

$$+\frac{(-1)^{\frac{n-1}{2}}}{\Gamma\frac{1}{2}}.\,\Gamma(\mu+2).\,\frac{\mathrm{P}_{-\frac{1}{2}}^{-\mu-\frac{3}{2}}(\cosh\alpha)}{\sinh^{\mu+\frac{3}{2}}\alpha}\Big\}\Big].$$

This we obtain by making use of (3.1) and the integral *

$$\int_0^\infty e^{-t\cosh\alpha}\,\mathrm{K}_\mu(t\overset{\sinh\alpha}{)}.\,t^{\mu-1}.\,dt$$

* Watson, *The theory of Bessel Functions*, p. 388.

$$= \sqrt{\tfrac{1}{2}\pi}\,.\,\Gamma(\mu-\nu)\,.\,\Gamma(\nu+\mu)\,.\,\frac{P^{\frac{1}{2}-\mu}_{\nu-\frac{1}{2}}(\cosh a)}{\sinh^{\mu-\frac{1}{2}} a}$$

$$R(\mu) > |R(\nu)| \text{ and } R(\cosh a) > -1$$

§ 4

Definite integrals involving $k_n(x)$.

Consider the integral $\int_0^\infty k_{2n}(x)x^m dx$ where n is $a + ve$ integer and $R(m+1) > 0$. Then

$$I = \int_0^\infty k_{2n}(x).x^m dx$$

$$= \frac{(-1)^n}{n!}\int_0^\infty x^{m+1}.\,e^x.\,\frac{d^n}{dx^n}\,[e^{-2x}.\,x^{n-1}]\,dx$$

$$\text{as } k_{2n}(x) = \frac{(-1)^n.\,x\,.\,e^x.}{n!}\,\frac{d^n}{dx^n}\,[e^{-2x}.\,x^{n-1}]$$

$$= \frac{1}{n!}\int_0^\infty e^{-2x}.\,x^{n-1}.\,\frac{d^n}{dx^n}\,\{x^{m+1}.\,e^x\}\,dx$$

$$= \frac{1}{n!}\int_0^\infty e^{-x}.\,x^{n-1}.\,[x^{m+1} + {}^nC_1.(m+1).\,x^m + {}^nC_2(m+1)m\,.\,x^{m-1}$$

$$+ \ldots + (m+1)m\ldots(m+2-n).\,x^{m+1-n}]dx$$

$$= \frac{1}{n!}\Big[\Gamma(m+n+1) + {}^nC_1.(m+1)\Gamma(m+n) + {}^nC_2.(m+1).\,m\times \tag{4.1}$$

$$\Gamma(n+m-1) + \ldots + (m+1)m \ldots\ldots (m+2-n)\Gamma(m+1)\Big]$$

From this we get when m and n are $+ve$ integers,

$$\int_0^\infty k_{2n}(x).\,x^m.\,dx \tag{4.11}$$

$$= \frac{1}{n!}\,[(m+n)! + {}^nC_1\cdot(m+1)(m+n-1)! + \ldots\ {}^nC_{m+1}(m+1)!(n-1)!]$$

$$\text{if } m+1 < n$$

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$$= \frac{1}{n!}\,[(m+n)! + {}^{n}c_1 \cdot (m+1)(m+n-1)! + \quad \dots \quad \dots$$
$$\{(m+1)m(m-1) \dots (m-n+2)\} m!] ;$$
$$\text{if } m+1 \geq n.$$

Also when $\mathrm{R}(m+1)>0$.

$$\int_0^\infty k_0(x)\cdot x^m \cdot dx = \int_0^\infty e^{-x}\cdot x^m \cdot dx = \Gamma(m+1).$$

Next consider the integral $\int_0^\infty e^{-ax}\cdot x^n \cdot k_{2n}(x)dx$ where $\mathrm{R}(a)>0$, $\mathrm{R}(m+1)>0$ and n is $a+ve$ integer.

Then

$$\int_0^\infty e^{-ax}\cdot x^m \cdot k_{2n}(x)\cdot dx \tag{4.12}$$

$$= \frac{(-1)^n}{n!}\int_0^\infty x^{m+1}\cdot e^{x(1-a)}\cdot \frac{d^n}{dx^n}\cdot[e^{-2x}\cdot x^{n-1}]dx$$

$$= \frac{1}{n!}\int_0^\infty e^{-2x}\cdot x^{n-1}\cdot \frac{d^n}{dx^n}\ \{x^{m+1}\cdot e^{x(1-a)}\}dx$$

$$= \frac{1}{n!}\int_0^\infty e^{-x(1+a)}\cdot\{x^{m+n}(1-a)^n + {}^{n}\mathrm{C}_1\cdot(m+1)(1-a)^{n-1}$$
$$+ \dots (m+1)\cdot m \dots (m+2-n)\cdot x^m]dx$$

$$= \frac{1}{n!}\left[\frac{(1-a)^n\cdot\ \Gamma(m+n+1)}{(1+a)^{m+n+1}} + {}^{n}\mathrm{C}_1\cdot\frac{(m+1)\cdot(1-a)^{n-1}}{(1+a)^{m+n}}\Gamma(m+n)+ \dots\right.$$

$$\left. + \dots \dots + (m+1\ m \dots (m+2-n)\ \frac{\Gamma(m+1)}{(1+a)^{m+1}}\right].$$

In particular if m is $a+ve$ integer and $\mathrm{R}(a)>0$.

$$(4.13)\ \int_0^\infty e^{-ax}\cdot x^m\cdot k_{2n}(x)\cdot dx = (-1)^m\cdot 2\cdot\ \frac{d^m}{da^m}\left\{\frac{1}{(1+a)^2}\left(\frac{1-a}{1+a}\right)^{n-1}\right\}$$

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$$=\frac{1}{n!}\left[\frac{(1-a)^n(m+n)!}{(1+a)^{m+n+1}}+\ldots+{}^nC_{m+1}\cdot\frac{(m+1)!(1-a)^{n-m-1}(n-1)!}{(1+a)^n}\right]$$

if $m+1<n$

and

$$=\frac{1}{n!}\left[\frac{(1-a)^n\cdot(m+n)!}{(1+a)^{m+n+1}}+\ldots+\frac{(m+1)m\cdots(m+2-n)m!}{(1+a)^{m+1}}\right]$$

if $m+1\geq n.$

Also if $R(a)>0$, and $R(m+1)>0$.

$$\int_0^\infty e^{-ax}\cdot k_0(x)\cdot x^m\cdot dx=\int_0^\infty e^{-(a+1)x}\cdot dx=\frac{\Gamma(m+1)}{(1+a)^{m+1}}$$

$$(4.14)\int_0^\infty e^{-x}k_n(x)\cdot x^\rho\cdot dx$$

$$=-\frac{1}{\pi.2^{\rho-1}}\cdot\sin\frac{n\pi}{2}\cdot\sum_{m=0}^{\rho}\frac{(-1)^m\cdot\Gamma(\rho+1)\cdot{}^\rho C_m}{(2m-n)(2m+2-n)}$$

This can be easily proved by making use of

$$\text{(i)}\int_0^\infty e^{-x}\cdot k_n(x)\cdot x^\rho dx=-\frac{1}{2^\rho}\cdot\frac{2}{\pi}\cdot\sin\frac{n\pi}{2}\times$$

$$\sum_{m=0}^{\infty}\int_0^\infty\frac{L_m(y)\cdot e^{-y}\cdot y^\rho dx}{(2m-n)(2m+2-n)}$$

and

$$\text{(ii)}\int_0^\infty L_m(x)\cdot e^{-x}x^\rho\cdot dx=0,\ m>\rho\text{ and }=(-1)^m\cdot\Gamma(\rho+1)^\rho C_m,\ m\leq\rho.$$

If $R(a)>0$ and $R\ b>0$ and n is $a+ve$ integer

$$\int_0^\infty e^{-ax}\cdot k_{2n}(bx)\cdot dx$$

$$=\frac{(-1)^n\cdot b^n}{n!}\int_0^\infty x\cdot e^{(b-a)x}\cdot\frac{d^n}{dx^n}\left[e^{-2bx}\cdot x^{n-1}\right]dx$$

$$= \frac{b^n}{n!}\int_0^\infty e^{-x(a+b)}\cdot\{x^n\cdot(b-a)^n+n\cdot x^{n-1}\cdot(b-a)^{n-1})\}dx$$

$$= b^n \left[\frac{(b-a)^n}{(a+b)^{n+1}} + \frac{(b-a)^{n-1}}{(b+a)^n}\right] = \frac{2b^{n+1}}{(b+a)^2}\cdot\left(\frac{b-a}{b+a}\right)^{n-1} \quad \ldots \quad \mathrm{I}$$

And again

$$\int_0^\infty e^{-ax}\cdot k_0(bx)dx = \frac{1}{a+b} \quad \ldots \quad \ldots \quad \ldots \quad \mathrm{II}$$

Putting in I, $a=ai$ we get when a and b are > 0,

$$\int_0^\infty \cos\, ax\cdot k_{2n}\cdot(bx)\cdot dx - i\int_0^\infty \sin\, ax\cdot k_{2n}(bx)dx$$

$$= \frac{2b^{n+1}}{(b+ai)^2}\left(\frac{b-ai}{b+ai}\right)^{n-1}$$

$$= 2b^{n+1}\cdot r^{2n}\cdot\frac{\cos 2n\alpha + i\sin 2n\alpha}{(b^2+a^2)^{n+1}}, \text{ where } \tan\alpha = -\frac{a}{b} \text{ and } r=\sqrt{a^2+b^2}$$

Hence we get when n is $a+ve$ integer and a and $b > 0$,

$$\int_0^\infty \cos\,(ax)\cdot k_{2n}(bx)dx = \frac{2b^{n+1}\cdot\cos 2n\left(\tan^{-1}\left(-\frac{a}{b}\right)\right)}{a^2+b^2} \tag{4.21}$$

$$\int_0^\infty \sin\, ax\cdot k_{2n}(bx)dx = -\frac{2b^{n+1}\sin\, 2n\left(\tan^{-1}\left(-\frac{a}{b}\right)\right)}{a^2+b^2} \tag{4.22}$$

While putting $a=ai$ in II we get

$$\int_0^\infty \cos\, ax\; k_0(bx)dx = \frac{b}{a^2+b^2} \tag{4.23}$$

$$\int_0^\infty \sin\, ax\cdot k_0(bx)dx = -\frac{a}{a^2+b^2} \tag{4.24}$$

$$(4.3) \quad \int_0^\infty e^{-ax} \begin{matrix}\cos\\ \sin\end{matrix}\, bx \cdot k_{2n}(x)\,dx$$

$$= \frac{2 \begin{matrix}\cos\\ \sin\end{matrix}\, 2n\left(\tan^{-1}\left(\frac{b}{1+a}\right)\right)}{\{(1+a)^2+b^2\}};$$

n being $a+ve$ integer, $a > 0$

$$(4.31) \quad \int_0^\infty e^{-ax} \cdot x^m \cdot \begin{matrix}\cos\\ \sin\end{matrix}\, bx \cdot k_{2n}(x)\,dx,$$

n and m being $+ve$ integers and $a > 0$

$$= 2 \cdot (-1)^m \cdot \frac{d^m}{da^m}\left[\frac{\begin{matrix}\cos\\ \sin\end{matrix}\, 2n\left(\tan^{-1}\left(\frac{b}{1+a}\right)\right)}{(1+a)^2+b^2}\right]$$

Similarly we can evaluate

$$\int_0^\infty \frac{e^{-ax} \cdot \sin mx \cdot k_{2n}(x)\,dx}{x} \text{ and} \tag{4.32}$$

$$\int_0^\infty \frac{e^{-ax} \cdot \sin mx \cdot k_{2n}(x)\,dx}{x^2} \tag{4.33}$$

Thus (4.32) and (4.33) can be obtained by integrating with respect to $a, \int_0^\infty e^{-ax} \cdot \sin mx \cdot k_{2n}(x)dx$ once and twice respectively between proper limits.

(4.31) can be evaluated for unrestricted values of n by first evaluating

$$\int_0^\infty e^{-z} \cdot L_n(z) \cdot \begin{matrix}\cos\\ \sin\end{matrix}\, (mz)\,dz.$$

§ 5

Some miscellaneous results regarding $k_n(x)$.

If m and s are $+ve$ integers (5.1)

$$\mathrm{I}=\int_0^\infty k_{2s}(x)\cdot e^{-x}\cdot \mathrm{L}_m^{(\alpha)}(2x)\,\frac{dx}{x}=0 \qquad \text{if } m<s$$

$$=(-1)^s\cdot\frac{1}{s}\begin{pmatrix}\alpha+m-s\\ m-s\end{pmatrix},\ m\geq s.$$

Proof :

$$\mathrm{I}=\int_0^\infty k_{2s}(x)\cdot(-1)^m\cdot\frac{dx}{x}\cdot\left[k_{2m}(x)-\binom{\alpha+1}{1}k_{2m-2}(x)+\ldots\right]$$

$$=0 \qquad \text{if } m<s$$

$$=(-1)^s\cdot\frac{1}{s}\begin{pmatrix}\alpha+m-s\\ m-s\end{pmatrix} \quad \text{if } m\geq s$$

$$\frac{1}{2}\int_0^\infty k_{2m}\left(\frac{x}{2}\right)\cdot \mathrm{J}_0(\sqrt{xy})\,dx \tag{5.11}$$

$$=\frac{1}{2}\int_0^\infty \Gamma(m+1)\,e^{-\frac{x}{2}}\cdot \mathrm{T}_0^m(x)\cdot \mathrm{J}_0(\sqrt{xy})\,dx$$

$$+\frac{1}{2}\,\Gamma m\cdot\int_0^\infty e^{-\frac{x}{2}}\cdot \mathrm{T}_0^{m-1}(x)\cdot \mathrm{J}_0(\sqrt{xy})\,dx$$

$$=\Gamma(m+1)(-1)^m\cdot e^{-\frac{1}{2}y}\cdot \mathrm{T}_0^m(y)+(-1)^{m-1}\cdot\Gamma m\cdot e^{-\frac{1}{2}y}\cdot \mathrm{T}_0^{m-1}(y)$$

$$=e^{-\frac{1}{2}y}\,\mathrm{L}_m(y)+e^{-\frac{1}{2}y}\cdot \mathrm{L}_{m-1}(y)=e^{-\frac{1}{2}y}\,[\mathrm{L}_m(y)+\mathrm{L}_{m-1}(y)]$$

This follows from Wilson's definite integral*

$$\frac{1}{2}\int_0^\infty e^{-\frac{1}{2}x}\cdot x^{\frac{1}{2}v}\,\mathrm{T}_v^n(x)\cdot \mathrm{J}_v(\sqrt{xy})\,dx$$

$$=(-1)^n\cdot e^{-\frac{1}{2}y}\cdot y^{\frac{1}{2}v}\cdot \mathrm{T}_v^n(y)\,; \qquad \{\mathrm{R}(v)>-1\}$$

* *Mess. of Math.*, Vol. LV, p. 158.

and the relations that

$$k_{2m}\left(\frac{x}{2}\right)=(-1)^m.\, e^{-\frac{x}{2}}\,[\mathrm{L}_m(x)-\mathrm{L}_{m-1}(x)]$$

$$=e^{-\frac{x}{2}}\,[\Gamma(m+1)\mathrm{T}_0^m(x)+\Gamma(m)\mathrm{T}_0^{m-1}(x)].$$

$$\int_0^\infty \frac{k_n(x)\cdot\cos x}{x}\,dx=\sum_{s=0}^\infty \int_0^\infty (-1)^s\;\frac{k_{4s}(x)\cdot k_n(x)}{x}\;dx \tag{5.12}$$

$$=\frac{2}{\pi}\sum_{s=0}^\infty (-1)^s\cdot\frac{1}{2s}\;\frac{\sin(4s-n)\frac{\pi}{2}}{4s-n}$$

Similarly

$$\int_0^\infty \frac{k_n(x)\cdot\,\sin x}{x}\,dx=\frac{2}{\pi}\sum_{s=0}^\infty \frac{(-1)^s}{(2s+1)}\;\frac{\sin(4s-n+2)\frac{\pi}{2}}{4s-n+2} \tag{5.13}$$

supposing both the series to be integrable term by term.

$$\sum_{n=0}^\infty \quad k_{2n}(x)\cdot\mathrm{P}_n\{\cos 2\theta\} \tag{5.2}$$

$$=\frac{1}{2\pi i}\int^{(0+)}\frac{dt.e^{x\left\{\frac{t^2-1}{t^2+1}\right\}}}{\sqrt{t^4-2t^2\cos 2\theta+1}}$$

just as $\sum_{n=0}^{s} \mathrm{J}_{2n+1}(z)\cdot\mathrm{P}_{2n}(\cos 2\theta)=$

$$\frac{1}{2\pi i}\int^{(0+)}\frac{exp.\left\{\frac{1}{2}\,z\left(t-\frac{1}{t}\right)\right\}dt}{\sqrt{t^4-2t^2\cos 2\theta+1}}$$

as given by Pincherle.

Again

$$(5.3)\;\sum_{n=0}^\infty (-1)^n\left[k_{2n}(x)-\binom{\alpha+1}{1}k_{2n-2}+\binom{\alpha+2}{2}k_{2n-4}-+\ldots\right]$$

$$=0\,;\;-1<\alpha<-\tfrac{1}{2}.$$

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This follows from the fact * that

$$\sum_{n=0}^{\infty} L_n^{(a)}(x)=0 \; ; \quad x>0, \; -1<a<-\tfrac{1}{2}$$

and that

$$(-1)^n \cdot e^{-x} \cdot L_n^{(a)}(2x)=k_{2n}(x)-\binom{a+1}{1}k_{2n-2}+ \quad \dots \quad \dots$$

(see B_1, p. 819).

It is known that $\int_0^\infty e^{-\xi} \cdot \xi^{2k} \cdot L_\mu^{(k)}(\xi) \cdot L_v^{(k)}(\xi) \cdot d\xi=0.$

But if m is a+ve integer

$$k_{2m}(x)=(-1)^m \cdot e^{-x}\Big[L_m^{(a)}(2x)-\binom{a+1}{1}L_{m-1}^{(a)}(2x)+ \quad \dots \quad \dots$$

$$(-1)^m\binom{a+1}{m}L_0^{(a)}(2x)\Big].$$

Therefore, n being $a+ve$ integer $> m$

$$\int_0^\infty k_{2m}(x) \cdot e^{-x} \cdot x^{2a} \cdot L_n^{(a)}(2x)dx=0, \tag{5.4}$$

Again † $h_v(x+y)=k_0(y) \cdot h_v(x)-k_2(y) \cdot h_{v+2}(x)+ \quad \dots \quad \dots$

so, if a is $+ve$

$$\int_0^\infty h_v(x+y) \cdot e^{-ay} \cdot dy=\frac{h_v(x)}{a+1}-\frac{2}{(1+a)^2}\sum_{m=1}^{\infty}(-1)^{m-1}h_{v+2m}^{(x)} \times$$

$$\times\left(\frac{1-a}{1+a}\right)^{m-1}.$$

* Einar Hille, *Proceedings of the National Academy of Sciences*, Vol. 12. "On Laguerre's Series," pp. 262-63.

† See B_2, p. 567.

$$= \frac{h_\upsilon(x)}{1+a} - \frac{2}{(1+a)^2} \sum_{m=1}^{\infty} (-1)^{m-1} \cdot h_{\upsilon+2m}(x) . \left(\frac{1-a}{1+a} \right)^{m-1} \quad (5.41)$$

$$\int_{-\infty}^{\infty} h_\upsilon(x+y) \cdot k_{2s}(y) \cdot \frac{dy}{y} = h_\upsilon(x) . \frac{(-1)^{s-1}}{s} + (-1)^s . h^{(x)}_{\upsilon+2s} . \frac{1}{s}$$

$$= \frac{(-1)^s}{s} \left[h^{(x)}_{\upsilon+2s} - h_s(x) \right] \quad (5.42)$$

This is analogous to (24), p. 565 of B_2.

$$*k_\upsilon(x+y) = k_0(y) \cdot k_\upsilon(x) + k_{\upsilon-2}(x) \cdot k_2(y) + \dots \quad x>0, \; y \geq 0.$$

Hence

$$\int_0^{\infty} k_\upsilon(x+y) \cdot k_{2s}(y) \cdot dy = \left[k_{\upsilon+2s}(x) + \frac{1}{2} k_{\upsilon+2s-2}(x) \right]$$

$$+ \frac{1}{2} k_{\upsilon+2s+2}(x) \quad (5.43)$$

if $s>0$

$$= \frac{1}{2} \left[k_\upsilon(x) + k_{\upsilon+2}(x) \right]$$

Similarly

$$\int_{-\infty}^{\infty} k_\upsilon(x+y) \cdot k_{2s}(x) \frac{dy}{y} = \frac{(-1)^{s-1}}{s} k_0(x) + \frac{1}{s} k_{\upsilon+2s}(x) \quad (5.44)$$

(5.43) and (5.44) are analogous to (24), p. 565 of B_2.

For $a > -1$

$$k_{2m}(x) = (-1)^m \cdot e^{-x} \cdot \Big[L_m^{(a)}(2x) - \binom{a+1}{1} L_{m-1}^{(a)}(2x) + \dots \dots$$

$$+ \dots \dots \dots + (-1)^m \binom{a+1}{m} L_0^{(a)}(2x) \Big].$$

* See B_2, p. 564.

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So,

$$\int_0^\infty k_{2m}(x)\cdot e^{-x}\cdot x^{\alpha}\cdot \mathrm{L}_n^{(\alpha)}(2x)\cdot dx\cdot 2^{\alpha} \tag{5.5}$$

$$=0,\ n>m$$

and

$$=(-1)^n\cdot\binom{n+\alpha}{n}\Gamma(\alpha+1),\ n\leq m$$

because

$$\int_0^\infty e^{-x}\cdot x^{\alpha}\cdot \mathrm{L}_m^{(\alpha)}(x)\cdot \mathrm{L}_n^{(\alpha)}(x)dx=\begin{cases}0 \text{ for } n>m,\\ \Gamma(\alpha+1)\binom{n+\alpha}{n} \text{ for } m=n\\ (m,n=0,1,2.\ \ldots)\end{cases}$$

(5.6) It is also easy to see by using the results, given by S. Goldstein,* involving $\mathrm{W}_{k,m}^{(x)}$ functions that the following are true

(i) $$\int_0^\infty k_{2n}^{(m)}(x)e^{-x}\cdot x^{n-1}dx=\frac{\Gamma n}{2^n},\ n \text{ being } +ve$$

(ii) $$\int_0^\infty x^{l-1}\cdot e^{-2(\alpha^2+\frac{1}{2})x}\cdot k_{2n}(x)\cdot dx=\frac{1}{2^l}\cdot\frac{\Gamma(l+1)\Gamma l}{\Gamma(1+n)\Gamma(l-n+1)}\cdot$$

$$\times\,{}_2\mathrm{F}_1(l+1,l,l-n+1,-\alpha^2);\ l>0,\ \mathrm{R}(\alpha^2+1)>0 \text{ and } |\mathrm{I}(\alpha)|<1$$

(iii) $$\int_0^\infty (2t)^{-\frac{1}{2}}\cdot e^{-t}\cdot \mathrm{J}_1(2^{\frac{3}{2}}t^{\frac{1}{2}})k_{2s+2}(t)dt=\frac{(-1)^s}{2e}$$

(iv) $$\int_0^\infty (2t)^{-\frac{1}{2}}\cdot e^{-t}\cdot \mathrm{I}_1(2^{\frac{3}{2}}\cdot t^{\frac{1}{2}})k_{2s+2}(t)dt=\frac{e}{2\Gamma(s+2)}$$

(v) $$\int_0^\infty t^{l-1}\cdot e^{-t}\cdot \mathrm{I}_v(2^{\frac{3}{2}}\cdot a\cdot t^{\frac{1}{2}})k_{2n}(t)\cdot dt$$

$$=\frac{\Gamma(\frac{v}{2}+l+1)\,\Gamma(\frac{v}{2}+l)}{\Gamma(n+1)\cdot 2^l}a^v\,{}_2\mathrm{F}_2\left(\frac{v}{2}+l+1,\frac{v}{2}+l;\ v+1,\right.$$

$$\left.\frac{v}{2}-n+1;\ a^2\right)$$

* The *Proc. L. M. Society*, Series 2, Vol. 34, part 2, pp. 103-125. "On the operational representation of Whittaker's confluent hypergeometric function and Weber's parabolic cylinder function."

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$$(vi)\ \frac{1}{s!}\int_0^\infty 2^l \,.\, t^{l-1} \,.\, (\Gamma\,\overline{s+2})^2 \,.\, k^2_{2s+2}(t)\,dt$$

$$=\Gamma\,(s+l+2)++\ \frac{l^2 \,.\, s}{1^1}\Gamma\,(s+l+1)+\frac{l^2(l-1)^2}{(2!)^2}s(s-1)\,\Gamma\,(l+s)+\cdots$$

$$(vii)\quad e\,.\,(2t)^{\frac{1}{2}} \,.\, e^{-t} \,.\, \mathrm{J}_1(2t)^{\frac{1}{2}} = \sum_{s=0}^{\infty}\frac{(-1)^s}{\Gamma\,(s+1)}\,k_{2s+2}(t).$$

In conclusion, I wish to express my best thanks to Prof. Ganesh Prasad, who first brought to my notice the papers of Prof. Bateman and who has been taking a very kind and keen interest in the preparation of this paper. My sincere thanks are also due to Prof. Bateman for his very valuable suggestions and remarks, which have been embodied in this paper.

Bull. Cal. Math. Soc., Vol. XXIV, No. 3 (1932).

Paper II

ON SOME RESULTS INVOLVING THE k-FUNCTION, A PARTICULAR CASE OF THE CONFLUENT HYPERGEOMETRIC FUNCTION.

By

N. G. SHABDE

In a previous paper* I have studied the k-function of Kármán. The object of the present paper is to add some more results involving these k-functions. The results obtained below are believed to be new. I wish to express my thanks to Prof. Ganesh Prasad for the kind interest he has taken in this paper.

§ 1.

$$x \sin x = k_2(x) - 3\, k_6(x) + 5\, k_{10}(x) - + \ldots\ldots$$

$$x \cos x = -2\, k_4(x) + 4\, k_8(x) - 6 k_{12}(x) + \ldots\ldots$$

Proof :—

From the first definitions of the functions we have

$$e^{ix\tan\theta} = k_0(x) + k_2(x)\, e^{2\theta i} + k_4(x) e^{4\theta i} + \ldots\ldots$$

Hence equating the real and imaginary parts

$$\cos (x \tan \theta) = \sum_{n=0}^{\infty} \cos 2n\theta .\, k_{2n}(x) \ldots\ldots \text{(i)}$$

$$\sin (x \tan\theta) = \sum_{n=0}^{\infty} \sin 2n\theta .\, k_{2n}(x) \ldots\ldots \text{(ii)}$$

Differentiating (i) with respect to θ we get

$$\sin (x \tan \theta) . x . \sec^2\theta = \sum_{n=0}^{\infty} 2n \sin 2n\theta .\, k_{2n}(x)$$

Putting in this $\theta = \frac{\pi}{4}$ we have

$$x \sin x = \sum_{n=0}^{\infty} n . \sin \frac{n\pi}{2} . k_{2n}(x)$$

* See a paper with the same title as of the present one in the *Bull. Calcutta Math. Society*, Vol XXIV, No 3, pp. 109–134.

$$= k_2(x) - 3k_6(x) + 5k_{10}'(x) - + \ldots\ldots$$

Similarly, differentiating (ii) with respect to θ we get

$$x \cos(x \tan\theta).\ \sec^2\theta = \sum_{n=0}^{\infty} 2n \cos 2n\theta \cdot k_{2n}(x)$$

Putting $\theta = \frac{\pi}{4}$ in this we have

$$x \cos. x = \sum_{n=0}^{\infty} n.\cos\frac{n\pi}{2} \cdot k_{2n}(x)$$

$$= -2k_4(x) + 4k_8(x) - 6k_{12}(x) + \ldots$$

§2.

$\sum_{n=0}^{\infty} (-t)^n.\ k_{2n}(x).\ k_{2n}(y)$ *can be expressed as a definite integral.*

Proof:—

We have* for $a > 0$ and n a positive integer

$$\int_0^{\infty} e^{-ax}.\ k_{2n}(x).\ dx = \frac{2}{1-a^2} \cdot \left(\frac{1-a}{1+a}\right)^n$$

$$= \frac{2}{(1+a)^2} \cdot \left(\frac{1-a}{1+a}\right)^{n-1}$$

Therefore, by an inversion formula

$$k_{2n}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{2}{(1-a^2)} e^{ax} \cdot \left(\frac{1-a}{1+a}\right)^n da \quad \binom{c>0}{x>0},$$

Let $\qquad \psi(x, t) = e^{-x.\frac{1+t}{1-t}} = \sum_{n=0}^{\infty} (-t)^n.\ k_{2n}(x)$

$$\text{If } \Omega = \Omega(x,y,t) = \sum_{n=0}^{\infty} (-t)^n \cdot k_{2n}(x) k_{2n}(y)$$

* H. Bateman : *Transactions of the American Mathematical Society,* Vol 33, No 4, pp. 817-831 (828).

$$\int_0^\infty e^{-ax}.\,\Omega.\,dx = \sum_{n=0}^{\infty} (-t)^n.\,k_{2n}(y).\int_0^\infty e^{-ax}.\,k_{2n}(x)dx$$

$$= \frac{2}{1-a^2}\sum_{n=0}^{\infty}\left(-t.\frac{1-a}{1+a}\right)^n k_{2n}(y)$$

$$= \frac{2}{1-a^2}.\,\psi\left(y,t.\frac{1-a}{1+a}\right)$$

Hence again inverting, we get

$$\Omega = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{2}{1-a^2}.\,e^{ax}.\,exp\left(-y.\frac{1+t.\frac{1-a}{1+a}}{1-t.\frac{1-a}{1+a}}\right)da$$

$$= \frac{1}{2\pi i}\,exp\left\{-(x+y)\,\frac{1-t}{1+t}\right\}\int_{k-i\infty}^{k+i\infty}\Big[exp\left\{xw-\frac{4yt}{(1+t)^2w}\right\}\times$$

$$\times\frac{dw}{1-\left(w-\frac{1-t}{1+t}\right)^2}\Big]$$

with $k>0$, by means of the *substitution* $a+\frac{1-t}{1+t}=w$

§3.

A. Milne obtains the following result* as regards the zeroes of $W_{k,m}(x)$:—

If $n=k-\frac{1}{2}-m$ vanishes, $W_{k,m}(z)$ has no zero apart from $z=0$. If k is increased by unity, another root is introduced. Now remembering that $k_{2n}(z)\,\Gamma(1+n)=W_{n,\frac{1}{2}}(2z)$ for $z>0$, we have the result that $k_2(x)$ ***has no zero apart from $x=0$ and each time n of $k_n(x)$ is increased by 2, another root is introduced.*** **The fact that $k_2(x)$ has $x=0$ as a root is obvious from the fact that $k_2(x)=2xe^{-x}$ when $x>0$.**

* The *Proceedings Edinburgh Math. Society* Vol XXXIII, 1914-15, pp.48-64 (51), "On the roots of the confluent hypergeometric functions".

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§4.

To evaluate $\int_0^\infty e^{(1-a)x} \cdot k_{2n}(x) . K_\nu\{(\sqrt{a^2-1})x\} x^\mu dx$

$a>1$, *n a positive integer and* $R\,(2+\mu) > |\,R(\nu)\,|$.

$$k_{2n}(x) = (-1)^n \cdot \frac{x \cdot e^x}{n!} \frac{d^n}{dx^n}\left[e^{-2x} . x^{n-1}\right]$$

$$= \frac{(-1)^n}{n!} \cdot \Big[(-2)^n . x^{n-1} + {}^nC_1 . (-2)^{n-1} (n-1) . x^{n-2} + {}^nC_2(-2)^{n-2} (n-1)(n-2) . x^{n-3} + \ldots\ldots + n(-2) . (n-1)!\Big] x e^{-x}$$

$$= \frac{(-1)^n . e^{-x}}{n!}\Big[(-2)^n . x^n + {}^nC_1(-2)^{n-1} . (n-1) x^{n-1} + \ldots (-2) \cdot x \cdot n \cdot (n-1)!\Big]$$

Hence the required integral

$$= \frac{(-1)^n}{n!}\Big[\int_0^\infty e^{-ax} . K_\nu\{(\sqrt{a^2-1})x\} x^\mu dx . \Big\{(-2)^n . x^n + {}^nC_1(-2)^{n-1}(n-1) . x^{n-1} + \ldots + (-2) x n (n-1)!\Big\}\Big]$$

$$= \frac{(-1)^n}{n!}\Big[(-2)^n . \frac{\sin (n+\mu)\pi}{\sin(\mu+n+\nu)\pi}\,\Gamma(\mu+n+1-\nu)\, Q^\nu_{\mu+n}(a)$$
$$+ (-2)^{n-1}(n-1)^nC_1 . \frac{\sin(n+\mu-1)\pi}{\sin(\mu+n+\nu-1)\pi}\,\Gamma(\mu+n-\nu)\, Q^\nu_{\mu+n-1}(a) + \cdot$$
$$+ \ldots + (-2)n(n-1)!\,\frac{\sin(\mu+1)\pi}{\sin(\mu+1+n)\pi}\,\Gamma(\mu+2-\nu)\, Q^\nu_{\mu+1}(a)\Big]$$

since * $\int_0^\infty e^{-t\cosh a} K_\nu(t \sinh a) . t^\mu \; dt$

* Watson: *Theory of Bessel functions,* p. 387

$$= \frac{\sin \mu\pi \,.\, \Gamma(\mu+1-\nu)}{\sin(\mu+\nu)\pi} \overset{\nu}{Q}_{\mu} (\cosh a).$$

§ 5.

To evaluate $\int_0^\infty e^{(1-a)x} k_{2n}(x).\, I_\nu\{(\sqrt{a^2-1})x\} x^\mu dx$

$a > 1$, *n a positive integer and* $R\,(\nu+\mu+1) > -1$.

Proceeding as in §4, the required integral is seen to be equal to

$$\frac{(-1)^n}{n\,!}\left[\int_0^\infty e^{-ax} I_\nu\{\sqrt{(a^2-1)}.\,x\} x^\mu dx \left\{(-2)^n x^n + \cdots \right.\right.$$

$$\left.\left. + n(-2)x(n-1)! \right\}\right]$$

$$= \frac{(-1)^n}{n\,!}\Big[(-2)^n \Gamma(n+\mu+\nu+1)\, P^{-\nu}_{n+\mu}(a) + (-2)^{n-1}.\,(n-1)^n C_1$$

$$\Gamma(n+\mu+\nu) \times P^{-\nu}_{n+\mu-1}(a) + \cdots + (-2)\, n(n-1)!\, \Gamma(\mu+\nu+2)\, P^{-\nu}_{\mu+1}(a)\Big]$$

since [*]

$$\int_0^\infty e^{-t\cosh a}\, I_\nu(t \sinh a).\, t^\mu dt$$

$$= \Gamma(\mu+\nu+1).\, P^{-\nu}_{\mu}(\cosh a),\ R(\mu+\nu) > -1$$

§ 6

To evaluate $\int_0^\infty e^{(1-a)x}\, x^\mu\, k_{2n}(x)\, J_m\{(\sqrt{1-a^2})x\} dx$

$1 > a > 0$ *and* μ, n *and m being positive integers,* $\mu+1 > m$

[*] Watson: l.c, p. 387.

Proceeding as in §4, the required integral will be seen to be equal to

$$\frac{(-1)^n}{n!}\left[\int_0^\infty e^{-ax} J_m\{(\sqrt{1-a^2})x\}\,dx.x\left\{(-2)^n x^n + {}^nC_1(-2)^{n-1}\times\right.\right.$$

$$\left.\left.\times(n-1)x^{n-1} + \cdots + \cdots + n(-2)x(n-1)!\right\}\right]$$

$$=\frac{(-1)^n}{n!}\Big[(-2)^n(\mu+n-m)!\,P^m{}_{\mu+n}(a) + {}^nC_1(-2)^{n-1}(n-1)\times$$

$$\times(\mu+n-m-1)!\,P^m{}_{\mu+n-1}(a) + \ldots + (-2)\cdot n\,(n-1)!\times$$

$$\times(\mu+1-m)!\,P^m{}_{\mu+1}(a)\Big]$$

Since

$$* P^m{}_n(\cos\theta) = \frac{1}{(n-m)!}\int_0^\infty \lambda^n \cdot e^{-\lambda\cos\theta}\,.\,J_m(\lambda\sin\theta)\,d\lambda$$

§ 7

The value of $\int_0^\infty e^{(1-a)x}\,.\,k_{2n}(x).\ J_m(bx)\cdot x^q\cdot dx$

$a>0$, R $(q+1+m)>-1$ and n a positive integer, can be at once written down by using the expansion of $k_{2n}(x)$ given in §4 and the † value of the Weber's integral $\int_0^\infty e^{-ax}\,.\,x^q.\ J_m(bx)dx.$

The value of the required integral will be seen to be equal to

$$\frac{(-1)^n}{n!}\Big[(-2)^n.\Big(\frac{b}{2}\Big)^m.\ (a^2+b^2)^{\frac{-(m+n+q)}{2}}\cdot\frac{\Pi(n+m+q)}{\Pi(m)}\times$$

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* Hobson : the *Proc. London Math. Society*. Series 1, Vol 25.

† Ganesh Prasad : *Spherical Harmonics*, part II, p. 239

$$\times F\left(\frac{m-n-q}{2}, \frac{m+n+q+1}{2}, m+1, \frac{b^2}{a^2+b^2}\right)$$

$$+\ +\ \dots\ +(-2)\,n\left(\frac{b}{2}\right)^m (a^2+b^2)\ \frac{-(m+q+1)}{2} \times$$

$$\times\frac{\Pi(m+q+1)}{\Pi(m)}\,F\left(\frac{m-q-1}{2}, \frac{m+q+2}{2}, 1+m, \frac{b^2}{a^2+b}\right)\Big]$$

For particular cases the hypergeometric function involved in the value of the integral can reduce to simple functions like Legendre's associated functions.

§8

$$\int_0^\infty e^{(1-a).x}\ k_{2m}(x).\ k_{2n}(x).\ x^\mu\ dx,\ a>0,$$

m,n and μ being positive integers can be evaluated by using the expansion of $k_{2n}(x)$ given in §4 and the value of the integral given in the result (4·13) on p. 125 of my previous paper.

The integral to be evaluated will be seen to be equal to

$$\frac{(-1)^n}{n!}\left[(-2)^n(-1)^n 2.\,\frac{d^{\mu+n}}{da^{\mu+n}}\left\{\frac{1}{(1+a)^2}\left(\frac{1-a}{1+a}\right)^{m-1}\right.\right.$$

$$+(-2)^{n-1}nC_1(n-1)\ (-1)^{n-1}2\times$$

$$\times\frac{d^{\mu+n+1}}{da^{\mu+n-1}}\left\{\frac{1}{(1+a)^2}\left(\frac{1-a}{1+a}\right)^{m-1}\right\}+\dots+(-2).2n(n-1)!$$

$$\left.\times\frac{d^{\mu+1}}{da^{\mu+1}}\left\{\frac{1}{(1+a)^2}\left(\frac{1-a}{1+a}\right)^{m-1}\right\}\right]$$